Platonism in the philosophy of mathematics

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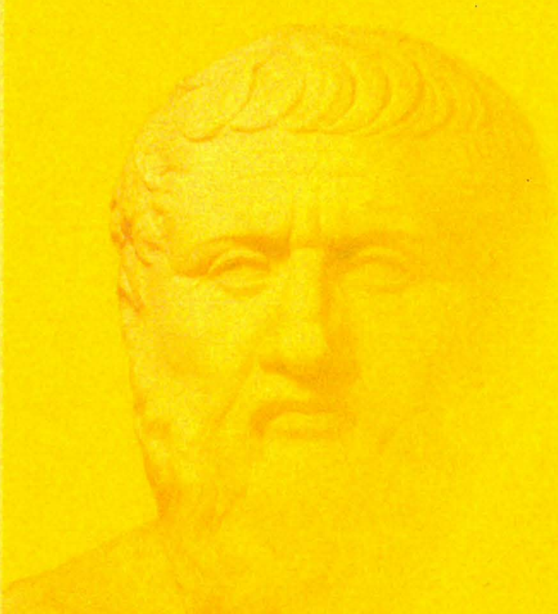

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Majda Trobok

PLATONISM IN THE PHILOSOPHY OF MATHEMATICS

Contents

Preface

In the dissertation for my Masters Degree, "Realism and Anti-realism in the Philosophy of Mathematics", I argued that the best answer to the main problems about mathematical truth is the platonist one.

However, there are many versions of platonism, and in this book, which is based on my PhD thesis, I explore the most prominent versions with a view to determining which of them should be preferred. To say that platonism is the best account of mathematical truth is not to say that it is without difficulties. And how these difficulties are dealt with by various versions of platonism is relevant to deciding which versions of platonism is best.

The book is divided into two parts. The aim of the first part is to give an argument for Platonism. The argument is in two main stages: I argue first for realism about mathematics, and then for the platonistic version of realism. The argument for the first stage is addressed to non-realist philosophers of mathematics, such as formalism, nominalism, and intuitionism. It si given in the first three chapters.

Chapter 1 shows that realism bout mathematics has two components. Chapter 2 argues for the first of them, realism about truth values, while Chapter 3 argues for the second component, realism about ontology.

In Chapter 4 I show why non-platonistic versions of realism about mathematics, and in particular, Maddy's "set-theoretic realism" are untenable.

The aim of the second part is to assess the virtues and vices of contemporary platonistic theories of mathematics. An introduction, Chapter 5, describes the main epistemological, semantic, and ontological problems that an adequate platonism must solve. Chapter 6 first details the wide variety of doctrines known as "structuralism", and then points out several difficulties with this doctrine. These include the relativity of the objects of a theory to the theory itself, and problems concerned with the notion of grasping a structure. The remaining chapters deal with Frege's logicism (Chapters 7 and 8), and then (Chapters 9 and 10) with the so-called "neo-logicism" and its attempt to retain the (supposed) insights of Frege's theory while avoiding its obvious pitfalls (including its inconsistency). The chapters in Frege's logicism are largely expository, but in those on neo-logicism I address the crucial features of the neo-logicist programme in depth. I conclude that neo-logicism is untenable.

The answer I give to the epistemological problems facing platonism is that, ultimately, it is the role of mathematics in natural sciences which gives us reason to believe mathematicians' claim about the abstract objects that are the concern of mathematics. This position allies me with the so-called "epistemological" holism of Quine. My answer to the semantic problem of indeterminacy is more tentative.

I summarise my views in the conclusion.

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I would also like to thank the University of Glasgow for allowing me to spend the academic year 1998/99 at their Department of Philosophy as a "Visiting scholar", as well as the Faculty of Arts and Sciences of Rijeka for allowing me to take the academic year off. During my stay at the University of Glasgow, Prof. Bob Hale and Dr. Philip Percival, were generous in helping me, and supervised my work. I would like to give them special thanks.

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1.

Introduction

1. Mathematics and philosophy

As even a cursory examination of the subject will illustrate, mathematics and philosophy have been related for centuries. Philosophical questions about mathematics were already posed by Pythagoras and his followers in the 6th century BC. However, questions of this nature were brought to the fore by Zeno of Elea half a century later, and similar questions have continued to exercise the greatest philosophical and mathematical minds ever since. A few examples will bear this out.

Zeno's reflections have had a lasting influence, and some of the issues he raised were not properly understood until the systematisation of analysis more than two thousand years later in the late nineteenth century. He presents several puzzles, called "antinomies", which make certain physical or mathematical results seem absurd. In one of the better known, "Achilles and the Tortoise", Zeno presents a powerful argument to the conclusion that Achilles cannot reach a tortoise he is chasing. His reasoning is as follows. Suppose that Achilles is one metre behind the tortoise. Even if we suppose that he is running ten times faster, by the time Achilles has reached the tortoise's starting point, the tortoise is 1/10 metres ahead. By the time Achilles reaches that point, the tortoise is 1/100 metres ahead, and so on *ad infinitum*. It follows that, beginning from a time at which Achilles is behind the tortoise, there is an infinite temporal series of increasingly later moments, such that at each moment in the series the tortoise is still ahead of Achilles. It follows, therefore, that to catch the tortoise Achilles must first perform each of these infinitely many tasks – traverse half the distance to the tortoise, and then half of that distance, and then half of the latter distance and so on, and then some more. But it seems to be in the nature of an infinite series that it must remain uncompleted; it would seem to be impossible to perform infinitely many successive tasks and then do something extra afterwards. So, seemingly, Achilles must remain behind the tortoise.[1]

Of course, we now know that Zeno's argument is based on the false mathematical assumption that an infinite series cannot have a finite sum, viz.

1 + 1/10 + 1/100 + ... = 10/9, and that Achilles would reach the tortoise after 10/9 metres. But it would be an anachronism to dismiss Zeno's antinomies because, viewed from the perspective of contemporary analysis, they involve simple fallacies. On the contrary, this perspective was only achieved as a result of long and intense mathematical labours which Zeno's antinomies helped stimulate and shape.

Almost two centuries after Zeno, Euclid wrote The Elements. This great work comprises thirteen books on plane geometry, the theory of numbers, irrationals and solid geometry. As such, it is a paradigm of pure mathematics. Yet it was inspired in part by Aristotle's logic and Plato's theory of Ideas, and it could not have taken the form it did without these influences. Consider the first definition (Book I), in which a point is characterised as something lacking dimensions. Euclid would not have found this definition acceptable if it had not been supported and sustained by Plato's ideas about perfect, non spatio-temporally located, mathematical objects. Prima facie, something without dimensions cannot be spatio-temporally located. So it seems that such a definition is only intelligible from within the framework provided by Plato's

[1] Aristotle formulates Zeno's four most famous paradoxes in his Physics, 239b, 8-33.

metaphysics. It therefore appears that philosophy and mathematics are so intertwined that the viability of certain philosophical presuppositions is a condition of the sense of some mathematical discourses.

The significance of philosophy for mathematics is equally evident when we turn to more recent developments in the twentieth century. Brouwer's philosophical view according to which mathematical objects are mental constructions lacking a truly mind-independent existence grounds his rejection of "classical" mathematics and logic and his subsequent development of "intuitionistic" alternatives to them.[2] Similarly, Gödel too claims that his philosophical convictions - namely that mathematical objects exist objectively - influenced his results in mathematical logic.[3]

[2] Since intuitionists hold that a mathematical object has no existence independently of procedures for "constructing" it, they are obliged to reject all mathematical techniques presupposing that the existence of mathematical objects is independent of our methods of identifying these objects. Such techniques have been taken to include even basic "classical" logical rules, like the principle of excluded middle and the rule of double negation. The principle of excluded middle - A or not-A - is untenable intuitionistically because, while its truth presupposes that one of A, and not A, is true, this cannot be guaranteed if the truth of A, and the truth of not-A require a construction of the kind given in a (constructive) proof: there is no guarantee that one of A and not-A is provable. A fortiori, the rule of double negation is not tenable intuitionistically either: without a guarantee that A or not-A, there is no justification for inferring from a proof of not not-A - which may merely have shown not-A to be contradictory - to A. Since classical mathematics frequently employs such "non-constructive" proofs - i.e. in which not-A is first shown to be contradictory, and the principle of excluded middle is then cited to infer A from not not-A, intuitionism requires a wholescale reconstruction of classical mathematics. Where non-constructive proofs cannot be replaced by constructive ones, intuitionistic mathematics simply rejects the results.

[3] In a letter to Hao Wang, Gödel writes:

> ... I may add that my objectivistic conception of mathematics and metamathematics in general, and of transfinite reasoning in particular, was fundamental also to my other work in logic. (letter of 7 December 1967)

In another letter to Wang, Gödel also suggests that the philosophical views of other mathematicians have influenced their work. He relates that, even though a paper by Skolem written in 1922 contained the core of the proof of the completeness of first

I will give just two more examples to illustrate the point I am making. The first concerns the acceptability of "impredicative" definitions in mathematics. A definition is "impredicative" if and only if it amounts to defining some entity by reference to a collection that contains the entity defined. A nice example is given by Berry's paradox. In Berry's paradox, the integer *n* is thus defined as: 'the least number not definable in fewer than nineteen syllables'.[4] The quoted words define *n* by citing a property which *n* uniquely has. This property involves quantification over a totality - the numbers not definable in fewer than nineteen syllables - to which *n* belongs. Of course, definition in this manner is a commonplace in classical mathematics. Yet, as Berry's paradox shows, it is implicated in some very puzzling phenomena. (The definition of *n* just given employs eighteen syllables.) And many mathematicians have responded to paradoxes like Berry's by refusing to countenance impredicative definitions on philosophical grounds. Henri Poincaré maintains that such definitions presuppose the independent existence of the defined entity and of the totality to which it belongs, and then rejects them on account of his philosophical view that mathematical objects do not exist independently of the mathematician. For

order logic as an almost trivial consequence of Skolem's results, Skolem did not draw this conclusion due to his philosophical views:

> I am still *perfectly convinced* that reluctance to use non-finitary concepts and arguments in metamathematics was the primary reason why the completeness proof was not given by Skolem or anybody else before my work.(letter of 7 March 1968)

(The inference Gödel calls 'almost trivial' was of course non-finitary, as is any completeness proof for the predicate calculus.)
See Gödel's letters to Hao Wang published in Wang (1974), pp. 8-11.

[4] Cf. Russell's gloss on Berry's paradox: 'the least integer not nameable in fewer than nineteen syllables' must denote a definite integer; in fact, it is 111777. But 'the least integer not nameable in fewer than nineteen syllables' is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction. (B. Russell, 'Mathematical Logic is Based on the Theory of Types', pp. 57-102 of Russell (1956).)

him, prior to the mathematical activity of defining a certain collection of mathematical objects, there is no such collection: to define a mathematical object is, literally, to construct it. It is therefore impossible to define an object by reference to a collection that the object would already be a member of if it existed: impredicative definitions are thence viciously circular.[5] By contrast, having opposing philosophical views as to the nature of mathematical objects and their relationship to us, Kurt Gödel sees no difficulty with impredicativity. He defends impredicative definitions by defending the philosophical view that mathematical objects do exist independently of the mathematician. Russell had responded to the paradoxes that beset mathematics at the end of the nineteenth century by imposing a strict requirement of predicativity in a "vicious circle principle":[6]

> If, provided a certain collection had a total, it would have members only definable in terms of that total, the said collection has no total

But in a famous article, Gödel responds as follows:[7]

> ... the vicious circle principle ... applies only if the entities involved are constructed by ourselves. In this case there must clearly exist a definition (namely the description of the construction) which does not refer to the totality to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it

[5] For more details about Poincaré's doctrine see e.g. Detlefsen (1990), pp. 502-509.

[6] Russell (1956), p.63.

[7] Gödel (1944) 'Russell's Mathematical Logic', in Benacerraf and Putnam (1983), p. 456.

> is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e., uniquely characterized) only by reference to this totality.
>
> ...Classes and concepts may, however, also be conceived as real objects ... existing independently of us and our definitions and constructions.

Let us turn to my final example, the axiom of choice. It says that for every set A of nonempty sets: $A = \{A_\lambda, \lambda \in J\}$, there exists a function $f : J \to \cup_{\lambda \in J} A_\lambda$, such that $f(\lambda) \in A_\lambda$ for every $\lambda \in J$.[8] In other words, for every set of nonempty sets there is a set comprising exactly one member of each the nonempty sets. This axiom had been used implicitly by many mathematicians in reasoning about various mathematical domains long before Zermelo explicitly formulated it in 1904.[9] Nevertheless, it is controversial. For its acceptability depends of our underlying philosophical views. If the underlying philosophical theory is one according to which mathematics is exclusively concerned with objects that can be constructed, or for which there is a method of construction, then the axiom of choice is untenable. For it is possible to construct a set A of nonempty sets such that it is not possible to determine the function f. This is why those who think that functions have no existence independently of us and our capacity to determine mathematical objects, such as Baire, Borel, and Lebegue,

[8] In symbols:
$(\forall\{A_\lambda\}_{\lambda\in J})\,((\forall\lambda)\,(\lambda\in J \Rightarrow A_\lambda\neq\varnothing) \Rightarrow (\exists f)\,(f\in(\cup_{\lambda\in J}A_\lambda)^J \wedge (\forall\lambda)\,(\lambda\in J \Rightarrow f(\lambda)\in A_\lambda)))$

[9] See Zermelo (1904) 'Proof that every set can be well-ordered', repr. in van Heijenoort (1967), pp. 139-141.

reject the axiom of choice, whereas others having no such qualms, such as Zermelo, Cantor, and Hilbert, accept it.[10]

What conclusion can be drawn from these examples? Clearly, this at least: it is impossible to deny or ignore the fact that throughout the history of mathematics, philosophical views have influenced not only mathematicians' reflective thought about their discipline, but their development and practice of it too. In this respect, nothing has changed since the days of Plato and Euclid. No less than their predecessors, contemporary mathematicians – indeed, all those who use mathematics (which basically means all of us) – are more or less explicitly, and more or less consciously, followers of certain philosophical theories. These philosophical theories are the fruit of the philosophy of mathematics. Their concerns are the latter's subject-matter.

The questions philosophy of mathematics raises, and attempts to answer, can be grouped into three fundamental categories:

(i) Metaphysical or ontological questions:

What is the subject matter of mathematics? Is it about creations of human mind or is mathematics about independently existing objects? What exactly are numbers, sets, functions, and so on?

(ii) Semantical questions:

[10] The fact that Hilbert accepted it might be surprising given that Hilbert was sceptical about the infinite and that the set A of nonempty sets usually *is* infinite. It might also seem that, on the assumption that Hilbert was not a platonist, it is not necessary to be a platonist to accept the axiom of choice and in general classical mathematics. The truth is that Hilbert introduced the non-finitar y elements of classical mathematics as "ideal" elements that were meaningless and whose role was just to preserve certain laws of classical logic and mathematics. So platonism conflicts with Hilbert's view on the *status* of infinitary mathematics. See Hilbert (1925) 'Über das Unendliche', Mathematische Annalen, 95, pp. 161-190, tr. as 'On the Infinite', in Benacerraf and Putnam (1983), pp. 183-201.
More about the axiom of choice can be found in e.g. Prijatelj (1980), pp. 122-151.

What does it mean to say that certain mathematical statements are true? What is the nature of mathematical truth? And what are the philosophical foundations of logic?

(iii) Epistemological questions:

How do we know anything about mathematics? Does our knowledge of mathematics involve experience and observation, or is it based on the exercise of some non-sensory, purely intellectual capacity?

Certain answers to these questions reflect a fundamental division of opinion between "realists" and "anti-realists". In the next section I will explain what their doctrines amount to.

2. The realism - anti-realism debate

The realism - anti-realism debate within the philosophy of mathematics illustrates a debate that arises in many areas of discourse. It is possible to be a realist in respect to any kind of thing, any kind of fact or any state of affairs: aesthetic properties, moral properties, universals, the spatio-temporally located objects, and so on. In general, to be a realist with respect to objects of a certain domain is to hold that these objects exist objectively, independently of us, our beliefs, constructions and our linguistic and cognitive practices, that our beliefs about these objects are *objectively* true or false, and, that, in the main, these beliefs are true.

By contrast, anti-realism with respect to such objects resists this characterisation of them.

Thus characterised, realism about some domain has two main components, and it is convenient to have separate terms for them. The first component of realism - the idea that the objects in question exist independently of our beliefs, conventions, our mathematical practice and so on - is often termed 'realism about ontology' or 'scientific realism' or 'internal realism'. I will ignore the latter terms, and call this component

simply "realism about ontology". The second component - the idea that statements about these objects are objectively true or false independently of our capacity to determine the truth values of these statements - is often called 'realism about truth value' or 'metaphysical realism' or 'external realism'.[11] Again, I will employ the former term, "realism about truth value", at the expense of the latter.

A moment's reflection reveals that combining "realism about ontology" with "realism about truth value" only yields "realism" as just characterised if "realism about truth value" is given a strong interpretation. Interpreted weakly, this doctrine simply says that statements regarding the objects in question are true or false objectively, and hence independently of our beliefs. But of course in this sense the doctrine is compatible with the supposition that *all* of our beliefs about these objects are false. By contrast, on a strong interpretation "realism about truth value" holds that statements regarding the objects in question are true or false, objectively, *in accordance (in the main) with our beliefs as to which they are.* Since "realism" was defined as incorporating the view that our beliefs about the objects in question are by and large true, "realism about truth value" must be given this strong interpretation if it is to combine with realism about ontology to yield realism *per se.*

Realism in the philosophy of mathematics

Realism in the philosophy of mathematics reflects the general characterisation of realism just given. When applied to the mathematical case, realism in ontology amounts to the idea that mathematical objects exist independently of our beliefs, conventions, our mathematical practice and so on, while, likewise, realism about truth value amounts to the idea that

[11] Brown (1999), pp. 149-150; Shapiro (1997), p. 37.

mathematical statements are objectively true or false, independently of our capacity to determine their truth value, and (on the strong interpretation), by and large in accordance with our beliefs. Unfortunately, terminology again varies, and some philosophers of mathematics, such as Dummett, reserve the term 'realism' for this latter view. Indeed, Dummett equates "realism" in the philosophy of mathematics with realism about truth value *in the weak sense*.

I find the identification of 'realism' with even the strong version of realism about truth value incongruous, and its identification with the weak version of realism about truth value especially so. Once we accept such classification, views that have been traditionally treated as anti-realistic turn out to be versions of realism. An example might be the nominalistic theory of Hartry Field[12], which holds mathematics to be objectively false. This theory is realist about truth value in the weak sense, and, hence, according to the classificatory scheme in question, realist. But very few philosophers of mathematics would of think of Field's theory as "realist". Indeed, its classification as such is contrary to common-usage (and hence common sense).

A similar point can be made against the proposal to identify "realism" in the philosophy of mathematics with realism about truth value under the strong interpretation, i.e. so that mathematical statements are not merely true or false objectively, but only false when we do not believe them true. One theory which is realist about truth value in this strong sense is so-called 'modal realism', the view that mathematics is about possibilities. According to this theory, mathematical statements do have objective truth value, and those statements we believe to be true are, by and large, indeed true, but mathematical objects as such do not exist. For mathematics is ex-

[12] See more about Field's nominalistic theory in Chapter 2 below.

pressible by means of statements containing a modal operator, and this operator undermines the ontological commitment to mathematical objects which ordinary mathematical statements appear to have. According to modal realism, the mathematical statements we take to be true are not about mathematical objects that exist; rather, they are true statements about possible objects, objects that might exist.[13] Again, although a view of this kind has a better claim to be labelled "realist" than does e.g. Field's mere endorsement of weak realism about truth value, it falls short of what most philosophers of mathematics would recognise as "realism" *per se*. After all, so to speak, although merely possible objects have enough reality for some statements about them to be true and other false, they do not quite have the reality of actual objects.

While less strong readings of the term 'realism' about mathematics have been employed, whereby it is not necessary to be realist in ontology to be labelled as realist, or, even, to be a strong realist about truth value – so that it is sufficient to be realist about truth-value in the weak sense – in this book I identity realism about mathematics as comprising realism about ontology, and realism about truth value in the strong sense.

Perhaps the most familiar realist view in the philosophy of mathematics, and the one this book is about, is platonism. What distinguishes platonism from realism in general is its specific version of realism about ontology. Platonism holds not merely that mathematics is about independently existing entities, but that these entities are abstract, in that they lack both spatio-temporal location and the "concrete" properties characteristic of spatiotemporally located items such as tables and rocks. Nevertheless, platonism holds the existence of the abstract, non-spatiotemporally located objects it identifies with mathematical objects to be strictly analogous to the existence of physical ob-

[13] See Hellman (1989).

jects. Even though mathematical objects are not spatio-temporally located, we do have the capacity to somehow grasp them.

Instead of identifying platonism with a species of realism, as I have, some classifications identify platonism with realism about ontology. Again, however, this usage is incongruous. Very few theorists would think of themselves as platonists unless they were prepared to assert realism about truth-value (in the strong sense) as well as realism in ontology.

It is clear that it is possible to endorse 'realism' in my sense without being a platonist. One illustration of this position is 'set-theoretic realism' endorsed by Penelope Maddy.[14] According to Maddy's version of realism, at least some mathematical objects, viz. certain sets, *are not* abstract; such objects are located in space and time. On her theory, when we perceive a physical object O, consciously or not we also perceive the singleton set[15] that has the perceived object O as its sole element. Numbers, on the other hand, are not objects but properties of sets.[16]

Anti-realism in the philosophy of mathematics

Anti-realism in the philosophy of mathematics denies realism. It denies that our mathematical beliefs are objective true beliefs concerning mathematical objects that exist independently of our constructions, intuitions, and mathematical practice. It can be divided into two main groups:

i) subjectivists, and

ii) eliminativists.

[14] See Maddy (1990).

[15] In general, "singleton S" is a set S^* whose only member is S.

[16] I will explain Maddy's view in more detail in Chapter 4. Maddy is often classified as being a platonist, and still further uses of 'platonism' can be found in the literature.

Subjectivists endorse the idea that mathematical objects, as well as facts, are subjective, that they depend on us. Let us mention here intuitionists, like Brouwer or Heyting, who maintain that mathematics is about constructions: mathematical objects do not exist independently of our minds. For a mathematical statement to be true amounts to a mathematical construction constituting a proof of it having been successfully realised (or at any rate, to the fact that a method for effecting a construction which will prove or disprove the statement has been demonstrated, and this method will in fact result in a proof of it). Subjectivism in this sense admits that most of the mathematical statements which classical mathematics takes to be theorems are indeed true. Nevertheless, it is revisionary of a significant body of classical mathematical practice, since this practice cannot be reconciled with the notion of truth just described.

In this respect, subjectivism contrasts with eliminativism. For eliminativists maintain that there are no significant mathematical truths to be apprehended: practically the whole of classical mathematics is literally false. To be sure, classical mathematics is applicable to the physical world, and therefore useful. But most of it is false nevertheless. Among radical eliminativists, like Hartry Field, the error of mathematicians' ways is ontological: while the platonist is right to suppose that mathematics (both classical and intuitionistic) is committed not only to the existence of mathematical objects, but to *abstract* objects, this commitment is completely misguided: (contemporary) nominalism is correct, in that *there are no abstract objects*.

By contrast, for "faint of heart" eliminativists, such as the "formalist" David Hilbert, mathematics goes wrong when it enters into the realm of the infinite. For Hilbert, finitary mathematical statements like: $2+2=4$, $3^2=9$, $4\neq5$, $2>2^{1/2}$, and suchlike, have an intuitively clear meaning. But statements like:

a+b=b+a, log(ab) = log(a) + log(b), and similar, are more problematic, and even, literally, senseless. In his view, statements of this kind function as ideal elements. As Hilbert says:

> to preserve the simple formal rules of ordinary Aristotelian logic, we must *supplement the finitary statements with ideal statements.*[17]

Although a great deal could be said regarding the articulation and development of anti-realist philosophies of mathematics, I will not be concerned with their intricacies. For my aim is to show that anti-realism about mathematics is fundamentally misguided. To this end, the remaining chapters of Part 1 present a positive argument for a specific version of realism - namely, platonism. Part 2 shows why the most important anti-realist criticisms of platonism fail, and selects one from among the many versions of platonism as the one to be preferred.

[17] See Hilbert, 'On the Infinite', in Benacerraf and Putnam (1983), p. 195.

PART 1

PLATONISM IN THE PHILOSOPHY OF MATHEMATICS

2

Realism (1) - Realism about Truth Value

Since platonism has been seen to be a version of realism (in the philosophy of mathematics), in this chapter and the next I aim to further clarify, and defend, (mathematical) realism. Realism was seen to comprise two theses – realism about truth value, and realism about ontology. I defend the former in this chapter; the latter in the next.

1. Truth aptness, and the notions of truth

In the philosophy of mathematics realism about truth value in the weak sense is the view according to which mathematical statements have objective truth values independently of our mind, language, or conventions; they are true or false independently of our capacity to determine their truth value. Realism about truth value in the strong sense supplements this thesis with the further claim that, in the main, our beliefs as to which truth values the various mathematical statements have are correct. In particular, it holds that mathematical theorems are not objectively false, but objectively true.

One might think that there is a great deal of redundancy in this characterisation. In particular, why should mathematical realism about truth value not be the view, simply, that mathematical statements are true or false? The answer to this question is the following one: if mathematical realism about truth value were characterised in this way, there would be different species of the doctrine that varied enormously in their import. For truth is not an unambiguous notion. There are various notions of truth, and if mathematical realism about truth value were characterised merely as the thesis that mathematical statements are true or false, what the thesis amounts to would then depend on the concept of truth employed.

To appreciate the difficulty, we must ask what it means to say of a statement that "it is true (or false)". In particular, what is it for a statement to be truth "apt" in the first place?[18] Expressions like "It is true that..." are used daily in a variety of different areas of discourse. But a closer analysis of the meaning of the term 'true' that occurs in them makes them look less trivial or self-evident and gives rise to many concerns.

Many of our utterances are not truth apt. An example might be "Shut the door!" or "What is the time?" and the like. So, it seems that utterances with imperative or interrogative force do not have truth values. But as soon as we move on from such statements to utterances that employ *declarative* sentences, the question of truth aptness is not as obvious as one might envisage. For example, we might happily say, "It is true that Santa Claus has got a long white beard", or "The truth is that Macbeth was a cold-blooded murderer". But do we really want to say that the notion of truth involved here is the same as the one involved when we say, "It is true that the nearest planet to the sun is Mercury?" Is the very same notion of true value applicable to statements concerning, for example fairy-tale characters, as the one applicable to statements about planets? It is hard to believe that it is. The domain of fictional individuals seems to be the polar opposite of the domain of physical objects, and hence of biology, or physics, or astronomy and the like. Accordingly, if the notion of truth applicable in the latter cases is the only one, we would have to revise our practice: in reality, nothing that concerns fairly-

[18] An expression is "truth apt" iff the concept of truth is applicable to it (obviously, e.g. a mere proper name, such as "Julius Caesar", is not truth apt).

tale or in general fictional characters is truth apt while everything that concerns science is.

Yet our application of truth and falsity to statements about fictional characters such as "The truth is that Lady Macbeth is a cold-blooded murderer" seems perfectly natural. Since this application is not odd or unacceptable, it seems preferable not to give the practice up. The right course appears to be, then, to suppose that a different notion of truth is employed in such cases from the one employed in e.g. the physical case.

A somewhat different case prompting the idea that there are different notions of truth is provided by statements which report rules or conventions. Consider a statement like "[for a woman] the perfect formula [for the social kiss is]: left, right but not left again and all with the minimum of human contact".[19] or "Castling is only allowed if neither rook nor king has moved". Both statements are truth-apt: one can say e.g. "It is true that castling is only allowed if neither rook nor king has moved". Yet, as in the case concerning fictional characters, it seems wrong to suppose that the notion of truth being employed here is the same as the notion of truth being employed in a statement like "Copper conducts electricity". Admittedly, the statements by which I illustrated conventions and rules have a reading on which a stronger notion of truth might be held applicable to them: the first might be construed as beginning "According to certain social conventions,...", while the second might be construed as beginning "In chess, the rules are such that...". Thus construed, these sentences are being employed to make statements about the conventions or rules which operate in certain locations, or in certain games. As such, they would appear to concern matters of fact, and a

[19] Morgan, John (2000) *Book of Modern Manners - Perfect Behaviour in an Imperfect World* (Harper Collins Publisher, London), pp. 15-16.

strong notion of truth would therefore seem to apply. However, they can also be construed not as reporting conventions or rules, but, rather, as *expressing* them. On this construal, a notion of truth still applies: one can, after all, endorse a convention or rule expressed by another who utters "S", and one way of so doing is by replying "It is true that S". However, the notion of truth that applies is not the strong one applicable to physical objects or their properties.

In general, the modern debate about truth suggests that in various cases akin to the ones just considered, various notions of truth of differing strengths can be apt, all of them falling short of the strongest notion paradigmatically applicable to statements about the physical world. In particular, we can have truth-aptness at a minimal level, and minimal standard of correctness with respect to statements of an area of discourse, in all those cases in which we really do not want to say the statements are about an external reality, but we do not want to give up saying that such statements can be true or false. Which areas of discourse are suited to be treated thus is of course controversial. Many hold that only weaker notions of truth can be applied to such 'problematic' domains as aesthetics, or ethics. Although we can certainly reply "That is true/false" in response to such statements as "Watermelon is delicious", "The painting 'La Gioconda' is beautiful", "Torture is wrong" etc., such statements do not seem to everyone to be objectively true. Of course, someone could endorse even a stronger notion of truth in the mentioned domains of discourse: they might say that fictional characters, or aesthetic or moral qualities, really exist. But this view is not of my concern here. Whatever its merits, the fact is that some have maintained that there are different notions of truth of varying strength, and that some areas of discourse in which the weaker notion is applicable are such that the strongest notion is not applicable. My concern is to clarify the realist thesis that mathematical statements are true or false independently of us and our practices, in the light of this fact.

In the light of the fact that different notions of truth have been hypothesised, what mathematical realism about truth value amounts to depends on the notion of truth employed. Obviously, we could understand "realism about truth value" to be neutral about the concept of truth employed: different species of mathematical realism about truth value would then arise depending on the notion of truth claimed to be applicable to mathematical statements. However, it is odd, historically, to call a position on which mathematical statements are minimally truth apt any kind of *realism* about truth value. In my terminology, then, "realism about truth value" is the thesis that mathematical statements are truth-apt with respect to the *strongest* notion of truth. But the strongest notion of truth applicable to an area of discourse is a notion of truth with respect to which the statements of that area are true or false independently of our thoughts, beliefs, and practices. That is why I have characterised "realism about truth value" in the philosophy of mathematics as the doctrine that mathematical statements are true or false independently of us, our mathematical thoughts and practices.

2. Two arguments for realism about truth value

Having said what realism in the philosophy of mathematics amounts to, the next task is to give reasons for endorsing such a view. There are two main arguments for the acceptance of realism: the 'obviousness' argument and the Quine-Putnam indispensability argument. Let us have a closer look at them.

The 'obviousness' argument

The obviousness argument is very brief. It is this: mathematical theorems are true in the strong sense, because their truth in the strong sense *is obvious*. This argument might seem trifling. However, the degree to which it impressed one of the greatest mathematicians of the twentieth century is striking.[20] We must therefore consider it further.

We certainly have the sense that there is no alternative to the answer "4" with respect to the question "2+2=?". The result is for us simply obvious. Even though we can imagine worlds quite different from the world out there, we certainly cannot imagine a result different from 4 of the operation "2+2", unless we change the meaning of '2', or '4', or the operation 'plus' or the meaning of 'equals'. Someone might reply that in certain books of, e.g. group theory, it is possible to find results like "2+2=1" so it is not true that just one result is possible. However, such equations are misleading, because the meaning of the term(s) employed is different from the usual one. First of all the domain is Z_3, so 'equals' means 'equals modulo 3' and the equation "2+2=1" means "$2+2\equiv 1 \pmod 3$" which is a different equation from "2+2=4"; viz. one calculation (as in "2+2=4") is the usual addition, while in the other ("2+2=1") the result (1) is obtained by determining the remainder of the division of 3 into the result of the usual addition (2+2), that is, into 4. It is therefore not the case that a different result of the same calculation is obtained; "2+2=4" is true and it is necessarily so.

Someone might complain that this argument is mind-centred and therefore not reliable. To retain certain results as obvious could just be due to the way our mind works; besides that there were results in the history of mathematics that seemed to be obvious but were not true. An example might be the claim that every continuous function on a closed interval is differentiable at all but a finite number of points in the interval. This claim had seemed obviously true to mathematicians until Weierstrass proved it not to be true by finding a

[20] It is given by Gödel (1944), pp. 447-69.

counterexample[21]. However, it might be said that the level of obviousness in this case is different from the level of obviousness in the case of "2+2=4". Moreover, one might reasonably say as much quite generally with respect to those (relatively rare) cases in which propositions which mathematicians' took for granted subsequently proved to be erroneous. After all, the concepts of continuous and differentiable functions were not well developed at the time.

One might worry that the obviousness argument is allied with an unduly restricted view of mathematics. For it suggests that mathematical theorems should be confined to those propositions that can be deduced from propositions – axioms one might think of them as – that are obvious. But the obviousness argument needn't be allied with this picture; quite the opposite. The emphasis on obviousness permits an "abductive" form of reasoning in mathematics in which "axioms" which are less than obvious are utilised on the grounds that they unify, and explain, theorems that we accept on the prior grounds of obviousness. Having argued that the elementary axioms of set theory are obvious, in that they 'force themselves upon as being true', Gödel himself makes this point when downplaying the significance of Cohen's proof of the independence of the Continuum Hypothesis from the standard axioms of set theory. *How*, one might ask, can the Continuum Hypothesis be objectively true, or false, if the axioms of ZFC[22] do not determine which it is? Faced with this

[21] In 1861 Weierstrass's result came as a shock to the mathematical community. He namely proved that for any constants a and b, such that $0<b<1$ and $ab>1+ 3\pi/2$, the function

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x)$$

is both continuous and not differentiable at any point of its domain.

[22] Let ZFC be the standard Zermelo-Fraenkel set theory including the Axiom of Choice.

question, Gödel warns us not to despair of resolving the status of the Continuum Hypothesis. There remains hope that we might supplement the standard axioms of set theory with others, *even if the supplementary axioms are less than obvious*:

> the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these 'sense perception' to be deduced...[23]

The simplicity of the argument from obviousness should not be allowed to detract from its impact; nor should it be thought to impose on mathematics an unduly restrictive methodology.

The Quine - Putnam indispensability argument

Self-evidence is not the only reason to consider mathematical theorems as being true where a richer notion of true is implied. Even though it is possible to imagine different physical theories about the real world or about worlds different from the one in which we live, all these physical theories are based upon the same mathematical apparatus. It seems that mathematical theorems have a particular status in our network of beliefs. They play a central role in our enterprise of understanding and controlling the physical world. Yet, unless mathematics is true, it is unclear what makes it useful in such applications. As Maddy points out

> Suppose, for example, that a physicist tests a hypothesis by using mathematics to derive an observational prediction. . . . what reason is there to take that observation to be a consequence of the hypothesis? And if it is not a consequence, it can hardly provide a fair test. In other words, if mathematics isn't true, we need an explanation of why it is all right to treat it as true when we use it in physical science.[24]

[23] Gödel (1944), p. 449.

Apart from that, in every branch of science: physics, biology, chemistry and so on, mathematics is a discipline *sine qua non* for developing any acceptable theory. The most disparate theories and the most disparate scientific domains about the physical world are based on the same mathematical apparatus: in genetic engineering or sociological statistical studies or astronomy or any other scientific topic mathematics is unavoidable and it is necessary to accept the same mathematical background. Mathematics is indispensable in science; that is why we are justified in asserting its being true. This is the "Quine - Putnam indispensability argument", also called the "Holism - Naturalism indispensability argument". The argument can be formulated in more detail as composed of three premises:

(i) Indispensability: mathematics is indispensable to natural science

(ii) Confirmational Holism: if the observational evidence supports a scientific theory, it supports the theoretical apparatus as a whole rather than some particular hypotheses.

(iii) Naturalism: natural science is for us the ultimate arbiter of truth and existence.

The argument then goes as follows: for (i), mathematics is indispensable to our theory of the world; by (ii), any evidence we have for the truth of some scientific theory is at the same time evidence for the truth of the mathematical apparatus employed in the formulation of the theory and in the deriva-

[24] Maddy (1990), p. 24.

tion from the theory of the predictions confirmation of which constitutes the evidence for the theory; therefore, by (iii), mathematics is true.

A weakness of this argument might be thought to be its third premise, Naturalism. While this premise assumes, in effect, that it is possible to obtain observational evidence for the literal truth of scientific theories, since the time of Berkeley at least, an instrumentalist tradition in the philosophy of science has denied that this is possible. However, while recent advances in science itself have made instrumentalism seem increasingly perverse (notwithstanding the somewhat heroic efforts of Bas van Fraassen to preserve the tradition), a looser version of the indispensability argument dispenses with this premise. This argument is sometimes called the "pragmatic" indispensability argument, in that it supports mathematical realism independently of scientific realism. It runs as follows:

(iv) many of the conclusions drawn from and within science could not be drawn without employing mathematical theorems.

(v) we are justified in drawing conclusions from and within science only if we are justified in taking the applied mathematics to be true.

(vi) irrespective of whether contemporary scientific theories are literally true, or reasonably thought to be literally true on the basis of observational evidence, we are justified in believing their observational predictions.

Clearly, these premises entail that we are justified in believing the mathematics employed in the derivation of observational predictions from contemporary scientific theories. Unlike the Quine-Putnam indispensability argument, this argument does not presuppose the truth or even the well-supportedness of scientific theories.

While the pragmatic version of the indispensability argument should allay the concerns of those with instrumentalist

leanings in the philosophy of science, there remains a radical critique of the argument which threatens to undermine it entirely; this has been given by Hartry Field[25]. Field accepts that *if* mathematics were indispensable to science, that would constitute very strong evidence for mathematical realism about truth value in the strong sense, and, hence, for the objective truth of mathematics. However, he denies the indispensability premise, and, hence, the antecedent to this conditional. In his view mathematics is not indispensable to science. Although mathematics plays a role in both the formulation of scientific theories, and in the derivation of consequences from them, it is possible both to reformulate those theories without employing mathematics, and to derive the same consequences from them without employing mathematics.[26]

Field's program – the project of demonstrating the theoretical (i.e. as opposed to the practical) eliminability of mathematics from contemporary science – is extremely ambitious, and the progress Field made in this respect cannot fail to impress. Upon reflection, however, it is the first part of the program which is really crucial. For the second part – the demonstration that mathematics does not allow the derivation of observational consequences from theories which are not derivable by mere logic – is supportable from any perspective. Practically everyone agrees that mathematics must be "conservative" in this sense. After all, if it were not, it would have *empirical consequences* (in combination with scientific theo-

[25] See Field (1980).

[26] Field maintains that mathematics is ontologically committed to abstract entities: the objective truth of mathematics requires the existence of abstract objects (numbers, sets and the like). Accordingly, he views the indispensability argument as an argument for platonism. As a nominalist, he holds mathematics to be a literally false, but useful fiction. However, these ontological issues are extraneous to the indispensability argument and its status. The argument is an argument for the objective truth of mathematical statements, whatever that should require. What the objective truth of mathematical statements would in fact require is a separate issue.

ries), and, hence, would itself be an empirical theory. But although Mill thought mathematics to have empirical content, no one else of any consequence has done so.

The first part of Field's program is a different matter. How, one might suppose, could one *begin* to formulate contemporary scientific theories – and in particular theories field theories such as Newtonian gravitational theory, without the aid of mathematics? Such theories are typically presented as families of mathematical models, each of the models being intended to represent a possible state of the world. However, Field's idea is that they might be presented, equivalently, as claims explicitly and solely about (the points in) space-time. Consider the theory of the (classical, mass zero) Klein-Gordon field.[27] This field is represented by a smooth scaler field φ: M$\rightarrow$R on Minkowski space-time (M,d) satisfying the field equation $\Box\varphi=0$. This much determines a set of mathematical models, each of which represents a dynamically possible history of the Klein-Gordon field. Now, the upshot of the Minkowski distance function, and the Klein-Gordon field, is to induce certain relations on (the points of) space-time. But Field's idea is to approach a theory of this kind by *beginning* with its import for relations on space-time. The *real* physical theory can be identified with some set of claims about primitive relations on the set of space-time points, and reformulated as such. A "representation" theorem can then be employed to show that these claims can be induced by the corresponding mathematical family of models (in this case, the models induced by the Minkowski distance function and the Klein-Gordon field).

In effect, the import of Field's technique is to exchange genuinely mathematical theorems about mathematical entities (models, and the domains of models), which are em-

[27] Here I am indebted to Malament (1982).

ployed to represent the physical world, for physical theorems which directly concern space-time, and hence the space-time points which compose it and relations on them. From a nominalist point of view, one might question whether this exchange involves much progress: is a space-time point really so different in nature from e.g. the null set construed as an abstract object? But the viewpoint peculiar to nominalism is not presently germane. The present concern is the question as to whether mathematical statements are indispensable in the formulation of contemporary theories of physics. And the technique just illustrated does seem to be successful in this respect, irrespective of the nature of space-time points. Space-time *is* the physical world. A theory solely concerned with it is not a mathematical theory. Accordingly, were Field's techniques generally applicable, so that all theories of contemporary physics could be reformulated along the lines he suggests, his case really would be made. The fundamental presupposition of the Quine-Putnam indispensability argument would be undermined. Mathematics would be dispensable after all.

However, even where it is applicable, Field's technique is less than completely successful. Some statements seemingly integral to the physical theory at issue resist reformulation.[28] And even more damagingly, Field's technique is heavily dependent on features which are peculiar to classical field theories. Field has not even begun to show how one might reformulate quantum-mechanical field theories, or alternatives such as Hamiltonian mechanics, or quantum mechanics.[29] Presently, therefore, the premises of the indispensability argument remain intact.

[28] Malament *op. cit.* gives such examples as the statements that "It is possible for the Klein-Gordon field to be nonconstant" and "The field evolves deterministically".

[29] See Malament, *op.cit.*, pp. 532-4.

3. Against mind-dependent truth

Even if the power of the argument from obviousness were conceded, mightn't it be objected that all the argument shows is that certain mathematical statements are true. It does not show they are *objectively* true? On the contrary, it might be felt that the very degree to which certain mathematical statements force themselves upon as being true (to evoke Gödel's phrase), tells against their objectivity. Is their non-objectivity not the best explanation of their powerful effect upon the mind? Isn't the best explanation of our certainty regarding fundamental mathematical theorems, and our inability to imagine their falsity, the Kantian supposition that the mind itself *constructs* their truth?

In recent times, the Kantian viewpoint is represented systematically by the school of mathematics known as "intuitionism". Following Brouwer, intuitionists maintain mathematical statements being true and mind-*dependent*, refuse certain laws of classical logic largely used in classical mathematics, such as the law of excluded middle. According to intuitionists, a (mathematical) statement is true if and only if a proof has been constructed or a method for proving it has been given. In this case it is not possible to assert that a statement is either true or false (independently of our knowing its truth value). A statement is neither true nor false prior to having a proof for it: it is not the case that for every statement S either S or not-S. A statement that is not true today might become true later, once we actually posses its proof. The law of excluded middle has therefore to be ruled out. If what is held is a mind-dependent conception of truth, then certain laws of classical logic have to be ruled out and classical mathematics cannot be accepted.[30]

[30] For a comprehensive discussion on intuitionism see Brouwer (1975) and Heyting (1956).

Faced with the opposing forces of intuitionism, the attempt to ground realism about mathematics on the argument from obviousness is unsatisfactory. Realism about mathematics requires the law of excluded middle. But it will hardly do to appeal to the obviousness of the law of excluded middle – one's opponents, the intuitionists, do not find this obvious at all. In fact they reject it. However, the indispensability argument is on firmer ground in this respect. Although the intuitionists have been remarkably successful in providing constructive proofs of much of classical mathematics, the fact remains that some parts of classical mathematics have resisted intuitionistically acceptable reconstruction. And some of them play a crucial role in science. Consider Gleason's theorem, which says that all probability measures over the projection lattice of the Hilbert space can be represented as density operators. This is a fundamental theorem in the foundations of quantum mechanics: indeed, it has a corollary the Kochen-Specker result that sharp values cannot be assigned to all variables in all states (and, hence, the refutation of certain "hidden variable" theories). But Gleason's theorem is essentially classical. Not only has no constructive proof of it been given, but no constructive proof of it seems possible.[31]

While the indispensability argument undoubtedly plays a crucial role in supporting realism about mathematical truth value, some realists give it an exclusive role. They do not see the importance of the 'obviousness' argument, and hold the argument that follows from applicability alone is enough for endorsing realism. Resnik for example, maintains that physical objects share the basic features of mathematical objects, which is enough 'to break down the epistemic and ontic bar-

[31] Hellman (1993) has argued that a constructive proof of Gleason's theorem is impossible.

riers between mathematics and the rest of science'[32]. Resnik's examples of such physical objects are quantum particles. And when he talks about reasons for endorsing mathematical realism he mentions firstly mathematical proofs for mathematical claims and secondly, the fact that the truth for mathematical theorems is confirmed in successful applications in science and everyday life and that it is necessary to presuppose the truth of mathematics in doing science.[33]

However, some realists, mostly platonists, are opposed to so great an emphasis on the indispensability argument. Their worry is that this argument assimilates mathematics too much too empirical science, and thereby fails to respect its *sui generis* nature. On balance, I side with them against Resnik. I think the 'obviousness argument' does have an important role to play. By definition, the indispensability argument can only claim that there is direct evidence for the truth of those mathematical theorems that play an indispensable role in the formulation and manipulation of scientific empirical theories. But only a small part of mathematics plays such a role in science. What then of the rest of it? No doubt some of it can then be supported indirectly, by means of the kind of "abductive" reasoning mentioned in the previous section: mathematical theorems T for which there is no direct evidence are supported indirectly because they follow from axioms which neatly unify and explain those theorems for which science provides direct evidence. But here judgements of "neatness" and "explanatory power" are essentially mathematical, and in any case one might expect some slack to remain, in that some

[32] See Resnik (1997), p. 101.

[33] This view is called pragmatic holism. He also mentions a third reason, viz. the results we get from comparing realism with the alternative positions in the philosophy of mathematics; realism 'accounts for mathematics as well or better than other contenders in the philosophy of mathematics' but this last argument is not of any interest in the present discussion. See Resnik (1997), p. 272.

mathematical theorems will fail even to fall into the category of theorems for which there is indirect evidence in this sense. The argument from obviousness might take up that slack.

4. Realism about truth value vs. realism about ontology

The arguments from obviousness and indispensability together make realism about truth value (in the strong sense) regarding mathematical theorems the most reasonable view. But this result falls short of realism about ontology, and hence, of realism itself. For, to establish that mathematical statements are true, objectively, is not in itself to say anything what it is in virtue of which mathematical statements are true. In particular, of itself, it is not to say that the truth of mathematical statements requires the existence of mathematical entities. The distance to be covered when moving from realism about truth value to realism about ontology might prove to be short. But it still has to be covered.

This fact is obscured by many discussions of the obviousness and indispensability arguments, and in particular by discussions of the latter. Historically, this has been because both the (foremost) proponents of the argument (Quine) and the foremost critics of it (Field), have accepted "conditional" platonism i.e. as captured in the conditional: if mathematics is true, abstract objects exist. So, for example, when advocating the indispensability argument Putnam writes:

> mathematics and physics are integrated in such a way that it is not possible to be a realist with respect to physical theory and a nominalist with respect to mathematical theory.[34]

[34] Putnam (1971), p. 57.

Similarly, in repudiating the indispensability argument, Field writes:

> although [mathematical entities] play a role in the powerful theories of modern physics, we give attractive reformulations of such theories in which mathematical entities play no role.[35]

And when expressing his nominalism he writes:

> Towards that part of mathematics which does contain references to (or quantification over) abstract entities - and this includes virtually all of conventional mathematics - I adopt a fictionalist attitude: that is, I see no reason to regard this part of mathematics as *true*.[36]

Nevertheless, the stature of these advocates of conditional platonism notwithstanding, the move from realism about truth value to realism about ontology must be examined with care; for several philosophers have tried to resist it.

[35] Field (1980), p. 8.

[36] Field (1980), p. 2.

3

Realism (2) - Realism about Ontology

In the previous chapter I gave reasons for endorsing realism about truth value in the strong sense: mathematical statements are apt for a strong notion of truth (as realism about truth value in the weak sense maintains), and, moreover, by and large, the mathematical statements we take to be true, *are* true. In this chapter I want to address the second element of realism about mathematics. This is realism about ontology. I shall argue that realism about ontology is much more intimately related to realism about truth value than one might at first suppose. Realism about ontology entails realism about truth value in the weak sense. And while the converse entailment does not hold – realism about truth value in the weak sense might be true when realism about ontology is false – realism about truth value in the strong sense does entail realism about ontology. Since reason has already been given for embracing realism about truth value in mathematics, this entailment completes the argument for realism about mathematics *per se*.

1. Realism about ontology

To be a realist regarding D's is to hold that D's exist objectively, in an external reality, independently of our constructions, thoughts, conventions or desires. D's could be a variety of quite different things: aesthetic values, quarks, tables and chairs, and so on. So, someone can be a realist with regard to a certain domain of discourse without being a realist with regard to a different one. For example, one might a realist about ontology with respect to unobservable theoretical entities in science such as quarks, while being an anti-realist about ontology with respect to aesthetic values.

In philosophy of mathematics, realism about ontology is the view according to which (at least some) mathematical objects exist, independently of our constructions and beliefs. For the realist about ontology in mathematics, then, mathematics is the science of certain objects: numbers, sets, functions, or other objects[37]. Mathematics is about such objects in the same way in which physical science is the study of physical objects, subatomic particles, and the like.

2. Realism about ontology entails realism about truth value in the weak sense

One might think that realism about truth value is independent of realism about ontology. Unlike the former, the latter makes not mention of truth at all. However, although there is no mention here of the notion of truth, realism about ontology is much more closely connected with realism about truth value than one might at first suppose. This is because the notion of truth is heavily implicated in what it means to say that objects of a certain kind "exist independently of our practices etc".

This is especially evident in the mathematical case. To say that there are mathematical objects with which mathematics is concerned is to say that singular terms in true mathematical statements typically refer.[38] But saying this much is not in itself sufficient for realism about ontology. After all, the intuitionist takes mathematical singular terms to refer too. So the

[37] If numerals are held to be abstract, causally inert objects then they are called types. If numerals are intended to be concrete, physical objects like signs of ink and the like, they are called tokens.

[38] This is not to say that *all* singular terms in true mathematical statements refer. The exceptions include occurrences of singular terms inside the scope of negation.

realist about ontology places a great deal of weight on the idea that mathematical singular terms refer not merely to objects, but *to objects, which exist independently of our mathematical practice*. But what does this difference – between the (intuitionist's) supposition that mathematical objects are mind-dependent, and the (platonist's) supposition that mathematical objects are mind-independent – really amount to? Dummett is surely right when he protests that the correct way to elucidate this issue is in terms of truth. The difference between the view that mathematical objects are mind-dependent, and the view that they are mind-independent, is the difference between the view that the truth values of statements about mathematical objects are somehow dependent, ultimately, on mathematical practice, and the view that whether or not a mathematical statement is true, or false, is entirely independent of our practice.[39] Accordingly, realism about ontology entails realism about truth value.[40] One cannot even express the doctrine without committing it to realism about truth value.

However, the realism about truth value that realism about ontology entails is only weak. The idea that the view that mathematical objects are mind-independent is to be explained in part as the view that mathematical statements are true, or false, objectively, and hence independently of our practice,

[39] See Dummett (1973) 'The Philosophical Basis of Intuitionistic Logic' in Dummett (1978), pp. 228-229.

[40] Someone might endorse the view that mathematical objects do exist but, since abstract and not spatio-temporally located, it is not possible for us to grasp them entirely and therefore the notion of truth we can apply is the one connected with our practice. According to such view, it is thence possible to assert something like: "As far as we know, _____ is true. And nothing more can be said about it". No notion of truth could be applied to mathematical statements apart from the one dependent of our practice. For the view that can be labelled as realism in ontology and anti-realism in truth value see Tennant (1987).
I find it difficult though, to see how, if we can refer to, and quantify over mind-independent objects, a mind-*independent* notion of truth cannot apply to statements about those objects, even though such a notion of truth might be epistemologically inaccessible in the sense that we might not know of such statements whether they are true or false.

falls short of the view that the statements we take to express mathematical theorems are by and large true. The following theses are jointly consistent: mathematical objects exist independent of our minds; mathematical statements about these objects are true, or false, independently of our mathematical practice; most of our mathematical beliefs about these objects are false. In practice, though, this conjunction of views would be rather strange. One would expect anyone sufficiently anti-sceptical to maintain that mathematical objects have a mind-independent existence to have little sympathy even for scepticism with respect to our beliefs about the properties of these objects.

3. Realism about truth value in the strong sense entails realism about ontology

The converse entailment, from realism about truth value to realism about ontology, is perhaps more controversial. Of course this entailment is a non-starter in the case of realism about truth value in the weak sense. As we observed in the previous chapter, some nominalists (such as H. Field), are openly realists about truth value in the weak sense without being realists about ontology. They maintain that mathematical statements are objectively true, or false, independently of our practice. But they do so because they think these statements are false. They maintain that these statements carry ontological commitments to abstract objects. But they also maintain that the commitments are not met. There are no abstract objects, and, hence, no mathematical objects either.

In contrast, the question as to whether realism about truth value in the strong sense entails realism about truth value is less straightforward. Again, some appear to have denied the entailment. Their grounds for so doing have been quite varied. All of their proposals turn on the idea that statements bearing apparent ontological commitments – whether in virtue of containing singular terms which appear to refer, or a prima facie existential quantifier "there are F's" – can be objectively true *even though those commitments are merely apparent*. There are three main variants of this strategy: Benacerraf's theory, the theory of substitutional quantifiers and the theory of "if-thenism". I will consider them in turn.

Benacerraf's theory. Benacerraf's early philosophy of mathematics is sometimes taken to be paradigmatic of the possibility of being a realist about truth value in the strong sense without being a realist about ontology. For Benacerraf denies that numerals refer, and therefore that "3" exists, whilst at the same time thinking that the statement "3 is a prime number" is nevertheless objectively true. He says:

> ... if the truth be known, there are no such things as numbers; which is not to say that there are not at least two prime numbers between 15 and 20.[41]

However, I think that an appeal to Benacerraf in this context is a confusion. According to Benacerraf, talking about numbers is a sort of shorthand. The latter sentence has not to be understood at face value. According to Benacerraf, saying that there are at least two prime numbers between 15 and 20 is to say that there is an abstract structure such that between the 15th and 20th places of the structure there are places with certain characteristics. Benacerraf's denial of the existence of numbers amounts to his contention that *any* object can play the role of *any* place in the structure.

However, whatever the merits of Benacerraf's "structuralist" view of number theory (see Chapter 6 below), and of his claims that this view precludes an ontology of numbers, it is a mistake to read Benacerraf as denying that mathematics has

[41] Benacerraf (1965), p. 294.

ontological commitments. The clear implication of his analysis is that number theory is committed to the existence of at least one complex structure. He denies the existence of numbers as objects but the statements whose truth value he asserts are not about numbers, that is objects in the first place. They are about places in an abstract structure. Benacerraf theory is therefore not a good example since he offers a different reading of certain propositions from what the standard, platonistic reading is. He does it in order to solve certain problems that apparently the existence of *numbers as objects* brings to surface while the existence of (abstract) structures does not. Benacerraf's theory therefore fails to indicate how realism in truth value without any objects whatsoever is possible if a stronger notion of truth is applied.

The theory of substitutional quantifiers. On the standard "objectual" reading of a statement involving the "existential" quantifier, a statement "there are F's", makes express commitment to the existence of F's. That is why it is called the "existential" quantifier. It follows that if the quantifier we employ e.g. in number theory when we say e.g. "there is a prime number between 15 and 20" is the standard objectual one, number theory is ontological committed to the existence of numbers. There is no room for argument about the matter. Equally, then, unless number theory is to be somehow reconstrued or "rewritten" in such a way that the theorem that there are prime numbers between 15 and 20 appears having an entirely different form, those who would deny it ontological commitment have no choice but to argue that the quantifier it employs is not the standard, objectual quantifier, but a different one.

What is characteristic of the objectual quantifiers is really a characteristic of the variables they bind: the variables bound by an objectual quantifier range over objects, so that a quantified statement is true if and only if the predicate is true of

some, or all (depending on the quantifier) of the objects over which the variables ranges. But apart from this interpretation, a "substitutional" reading of the quantifiers is possible too. On this interpretation, the truth conditions for the substitutional quantifier are quite different. They are given in terms of substitution instances. For example '$\exists x\ Fx$' is true if and only if, for some *singular term* "a", Fa is true. It is thence possible to have a statement false on the objectual reading, while true on the substitutional reading. An example might be the sentence "Something x is such that x is a winged horse". It is false on the objectual reading, since the objects over which the variable the quantifier binds ranges do not include among their number a winged horse: there is no existing object such that it is a horse and has wings. On the substitutional reading it is true: if we insert the singular term "Pegasus" in the phrase "x is a winged horse", the result – the statement "Pegasus is a winged worse" – is true. The idea, then, is that the quantifiers employed in number theory might be substitutional quantifiers. In that case, the truth of the statement "there are prime numbers between 15 and 20" would not *explicitly* require the existence of an object which is a number, which is prime, and which is between 15 and 20. It would require no more than that for some numeral n, the statement "n is a prime number between 15 and 20" is true.

Well, the response to this proposal is obvious. Sure, there is now no longer any explicit ontological commitment carried by the quantifier: we are not explicitly supposing the variable in "For some x, is a prime number..." to range over objects. But there are implicit commitments nevertheless.

Firstly, the substitutional reading of the quantifier cannot simply take substitution instances – such as, in the case of number theory, numerals – for granted. What does "There is an x such that x is numeral such that..." mean? If the quantifier is objectual, the variable ranges over objects. But then numerals have to be thought of as abstract objects. If they are not, a number-theoretic statement to the effect that there is some very large number with certain properties would turn out false simply because no one has every written down a numeral which refers to that number. So the ontological commitments of number theory – indeed, commitment to abstract objects – is restored. And how else can talk of substitution instances be construed, if not by means of the objectual quantifier?

Secondly, how can a substitution instance, such as "17 is a prime number between 15 and 20" be true unless the substituted name has a reference? Remember, the notion of truth at issue here is a strong one. It is supposed both that the truth of this statement is independent of mathematical practice etc., and that it is known to be true: the object of discussion is the consequence for ontology of realism about truth value in the strong sense. I do not think that the statement "Pegasus is a winged horse" can be true *in this sense* unless the name "Pegasus" really does refer. To say "Pegasus is a winged horse" is true on the substitutional reading is to apply a weaker notion of true in the first place.

The theory of "if-thenism". There is one more option left for those convinced that realism about truth value does not necessarily imply any ontological commitment: the modal option. The idea is that mathematical statements are, in effect, about *possibilities.* This idea has various manifestations. If, to return to the sort of viewpoint illustrated by Benacerraf, it takes such a form as to reconstrue e.g. "there is a prime number between 15 and 20" as a statement "there is a possible structure, in which there is a place between the 15th and 20th having such and such characteristics", then the ontological commitment of the statement remains explicit: it is not a commitment to actual objects, but it is a commitment to possible ones.

A better alternative, then, is for the modalist to resort to "if-thenism", i.e. the view according to which apparent ontological commitments are removed by putting them into the antecedent of a conditional. This alternative might formulate Benacerraf's view as follows "*If* there is a structure meeting certain conditions (such as those imposed by the Peano axioms), then it is true of the structure that between its 15th and 20th place ...". This alternative offers better prospects of success. Statements of this form might well be objectively true, independently of us, even though no structures or any other mathematical objects exist. However, even though it is a coherent view, it is a very implausible reconstruction. It does not reflect mathematical practice. In number theory, for example, we do not employ variables over structures. Rather we employ numerals that seem to function as singular terms, and variables, which appear to range over numbers.

Of course, there are many mathematical domains in which we assert "if-then" theorems. For example, in group theory we might assert that: "If a set with an operation on it is a group and its power is a prime number then such a set is an Abelian group". Examples like this are legion. But they do not cover the whole all mathematics. Let us take the axiom of Infinity: it asserts that there *exists* an infinite set; it is not clear what form of "if-theism" should be applied in these cases. Apart from that, if-thenism does not explain how come mathematics is successful and unavoidable in science, given that it is just about possibilities.

4. The theory of truth

We have just seen that attempts to resist the implication from realism about truth value in the strong sense to realism about ontology are less than successful. But the best way to bring out the unavoidability of this implication is to reflect further on the strong notion of truth that realism about truth value involves.

Since the idea of a strong notion of truth in this sense is that truth depends upon features of an objectively existing world, the requested theory of truth must involve an ontology of facts. Besides that, since it is held that a mathematical statement is true or false independently of our capacity to determine their truth value, the endorsed theory of truth cannot be epistemic. Namely, the epistemic conception means that a sentence S is true iff ______, where the blank contains an epistemic condition on S, such as: being verified, justified or warranted so it depends on our being epistemologically successful.

The theory that seems to fulfil all the requested conditions is the correspondence theory of truth.[42] Since there is no general agreement as to what a correspondence theory of truth amounts to, I shall briefly sketch the version of the theory I advocate. On my reading, the correspondence conception of truth comprises the disquotational biconditional for truth joined with a disquotationally defined reference relation.

Since Tarski's results on defining truth, almost everyone interested in truth theories has held that in order to be acceptable a truth theory must imply a disquotational biconditional for truth. That is, it must imply a sentence of the form

[42] Some philosophers do not believe that realism require correspondence truth. To cite Resnik who says that

> Some philosophers believe that realism is committed to the view that truth depends upon features of the world 'out there'. If in the mathematical case this just means that our theories are truth or false independently of out proofs and constructions, then it can be accommodated using an immanent, logical conception of truth. [Resnik (1997), page 32]

(Disquot. T): 'p' is true if and only if p, where the letter 'p' is a schematic letter standing in place of sentences. An example of the first instance might be simply:

'Grass is green' if and only if grass is green.

However, this biconditional is not enough since, as yet it is neutral as to how the right side of the sentence is to be construed.

To avoid formal complications and inconsistencies, we cannot simply postulate all these biconditionals. We need to derive them in a formal theory. And to do that we need a reference relation as well: we need to suppose that the singular terms and predicates of the language do refer. It is with this supposition that truth abandons ontological neutrality. In one sense, the reference relation can be disquotational too. It should obey:

(Disquot. Sat): x satisfies 'F' if and only if x is F;

(Disquot. Des): 't' designates x if and only if t=x.[43]

However, the disquotationally defined reference relation is a word-world relation. It relates words to objects. It follows that examples like: 'Hamlet' refers to Hamlet, are not disquotationally defined reference relations since they are not *world*-word relations. According to Disquot. Des., for "Hamlet" to refer to Hamlet, there must be an x to which Hamlet is identical. But in the absence of an ontology of fictional objects, there is no object x which is identical to Hamlet.

The strong, ontologically committed notion of truth I have adopted here is not incompatible with other features of truth that so-called "deflationists" have emphasised. Statements of

[43] According to Resnik, such a conception makes room for our most fundamental realist intuitions by permitting truth to be independent of our present theories and methods; see Resnik (1997), p. 15.

truth and falsity can still sometimes be used as linguistic short cuts for talking about reality by referring to sentences. As Quine rightly says, 'the truth predicate can serve as a device for switching from talk of reality to talk of sentences'. If I say "What the Peano axioms assert about the natural number sequence is true", the truth predicate works as an abbreviation with which I assert that 1 is a natural number, and that: if a is a natural number then a^+ (its immediate successor) is a natural number, and so on through the five Peano axioms.

4

Platonism

I have defended realism in the philosophy of mathematics, and in particular realism about ontology. So mathematics concerns itself with certain objects - numbers, sets, functions, groups etc. – and the claims it makes about these objects are determinately and objectively true or false. A question naturally arises at this point. Where are these objects, and what sorts of things are they? Different answers to this question reflect different versions of realism. The most basic division amongst them is one between so called 'faint of heart' realism[44], and platonism. My aim in this chapter is to explain and defend platonist realism against the faint of heart alternative. In effect, I defend it by rejecting the faint of heart realist's answer to the question just asked. I begin by characterising, and then attacking, one of the most appealing contemporary formulations of this answer, Maddy's 'set-theoretic' realism.[45] I end by showing that, quite generally, faint of heart realism is intrinsically flawed.

1. 'Faint of heart' realism

'Faint of heart' realism opposes the platonist version of realism I wish to defend by denying that mathematical objects are abstract objects: it holds that at least some mathematical objects are *concrete* objects that are part of the spatio-temporal world. I begin my critique of this doctrine by examining one influential version of it, viz. Penelope Maddy's 'set-theoretic realism'.[46]

[44] For the terminology see Brown (1990).

[45] Maddy is often, I would say wrongly, classified as being a platonist, and still further uses of 'platonism' can be found in the literature.

[46] Maddy (1990).

Penelope Maddy's 'set-theoretic realism'

The 'set theoretic' realism Maddy endorses is the theory according to which some sets are concrete objects located in space and time. This radical view of sets brings them – and hence brings mathematical objects – 'into the world we know and into contact with our familiar cognitive apparatus'[47]. According to her, the ontology and epistemology of mathematics is the same as that of other sciences. Set theory is about sets and their properties in the same way in which physics is about physical objects and their properties, and we grasp sets in pretty much the same way in which we see physical objects. To illustrate what she has in mind, let us consider her own example. The example is about Steve who needs two eggs for a certain recipe, and opens the fridge door. He finds there an egg carton and sees three eggs in it. According to Maddy, he does not just see three eggs. He sees something more:

> My claim is that Steve has perceived a *set* of three eggs. By the account of perception just canvassed, this requires that there be a set of three eggs in the carton, that Steve acquire perceptual beliefs about it, and that the set of eggs participate in the generation of these perceptual beliefs in the same way that my hand participates in the generation of my belief that there is a hand before me when I look at it in good light.[48] (my emphasis)

On Maddy's account, Steve can see not just the three eggs, but the *set* of three eggs too, because like the eggs themselves that set is spatiotemporally located. It is located in the same place in which its elements, the three eggs, are located. By generalising from this example, Maddy embraces an extreme version of set-theoretic realism. She holds that *all* sets which contain only physical objects, and whose members contain only physical objects, and whose members of whose members contain only physical objects, etc., are located in space and time. More precisely, sets in which physical objects, and only physical objects, are implicated in this way, are located where those physical objects are located. For example, a set of higher order, like the set whose two members are the set of the three eggs and the set of Steve's two hands, is likewise located where its members are located. So, since the set of three eggs and the set of Steve's hands are located where the three eggs and Steve's hands are, this higher order set is located there too. The same holds true even in the case of extremely complicated sets in which all and only physical objects are implicated. Indeed, Maddy generalises further on the basis of the physicalist doctrine that everything is physical. For she conjoins physicalism with the considerations just adduced to derive the conclusion that *all* sets, without exception, are located where the physical objects that are implicated in them are located.

[47] Maddy (1990), p. 48.

[48] Maddy (1990), p. 58.

But what, one might ask, about "pure" sets (i.e. the sets built up from the null set). Unlike the strictly impure sets built up from physical objects, neither the empty set, nor, hence, the pure sets built up from it in standard set theory, can be located anywhere in space and time. For Maddy, this is just to say that the empty set and the pure sets generated from it do not exist. She says:

> the pure sets aren't really needed. The set theoretic realist who would simultaneously embrace physicalism can take the subject matter of set theoretic science to be the radically impure hierarchy generated from the set of physical individuals by the usual power set operation, except that the empty set is omitted at each

> stage.... So the set theoretic realist can locate all the sets she needs in space and time.[49]

In addition to construing the strictly impure sets as physical objects, and locating them in physical space, Maddy goes even further in the case of the singleton sets whose single members are physical objects. Where x is a physical object, she does not merely locate singleton x ({x}, i.e. the set whose only member is x) where x is located. She actually *identifies* singleton x with x itself:

> we take it that the physical objects, x, the individuals from which the generation of the iterative hierarchy begins, are such that x={x}.[50]

In sum, then, Maddy's set theoretic ontology is characterised by two modifications. She namely says:

> I've suggested two minor alterations in the set theoretic realist's ontology: the identification of physical objects with their singletons and the elimination of pure sets.[51]

Accordingly, Maddy maintains that pure sets are not necessary to get the Zermelo (or von Neumann) ordinals, or to accept a set theory strong enough to perform the mathematical tasks standard set theory performs. In particular, the existence of two physical objects x and y enables the ordinals to be generated in the following way: x, {x, y}, {x, y, {x, y}} and so on. On the other hand, on purely pragmatic grounds Maddy is prepared to relax her restriction of set theory to the strictly impure sets built up from physical objects. She thinks that for

[49] Maddy (1990), pp. 156-7.

[50] Maddy (1990), p. 153.

[51] Maddy (1990), p. 157.

practical reasons only, it is best to keep the empty set, which she treats as a notational convenience.[52]

Finally, what are numbers in this theory? According to Maddy, numbers are not sets. If they were sets, it would be possible to identify what sets they are, which it is not.[53] Numbers cannot be any other objects either. If, Maddy argues, the number 5 were an object, this object would have (outside the natural number sequence) certain properties that are not relevant for the numerical functioning of such object. Since there are no arguments according to which such properties could be identified, the number 5 cannot be an object. The conclusion is that numbers are not objects. What are numbers then? They are properties of sets:

> for the set theoretic realist, sets have number properties in the same sense that physical objects have length.[54]

As we compare different lengths we can also compare sets according to their "size"; numbers are sets' properties, analogously to physical properties.[55] We grasp the "measure" of a set - that is the number of its element when grasping the set itself - in the same way in which we grasp the physical properties of a physical object when grasping the object itself. Numbers are not included in the set-theoretic ontology since, as Maddy holds, there is nothing of mathematical relevance in number-theory that cannot be expressed without explicit

[52] Maddy (1990), p. 157, footnote 10.

[53] There are several possible reductions of numbers to set theory and no mathematical result can sort out which reduction is the right one. See more about this problem in the next chapter.

[54] Maddy (1990), p. 98.

[55] The only disanalogy consists in the fact that it is not possible to "measure" sets with different scales, which is possible when measuring the length, mass, density or suchlike.

reference to numbers. All we can say about numbers can be said by using the von Neumann's (or Zermelo's or some other) ordinals; for example: '2 is prime' says 'if x is equinumerous with {{}, {{}}}, then there are not two sets of cardinality less than {{}, {{}}} but greater than {{}} whose cross product is equinumerous with x'.[56]

Critique of Maddy's 'set-theoretic' realism

Maddy maintains that it is possible to construe the empty set as a mere device which, though convenient in practice, is in principle dispensable, while at the same time identifying each physical object x with its singleton set, and maintaining the physicalist doctrine that all things are physical. But is this really possible? Trivially, it is impossible if there is only one physical object. For in that case the set-theoretic hierarchy collapses immediately. The starting point, given a physical object x, is that x = {x}. But if the physical object x is identical to singleton x, then the set whose only member is singleton x – viz. {{x}}- must be identical to singleton x. Let us take the set {{x}} in which x is an - apple. Its only element is the set {x} which is identical with x, which means that {{x}}={x}, and that means (since {x}=x) that {{x}}=x and so on for all the others ordinals. So, the identity does preclude von Neumann's (or Zermelo's) reduction of the ordinals.

This point is not lost on Maddy. Indeed, she herself observes that at least two physical objects are needed if her set-theoretic realism is to be viable. With two individuals x and y, she says 'a version of the ordinals can be constructed without pure sets - x, {x, y}, {x, y, {x, y}}, and so on'[57]. Of course, Maddy is right in this. However, that she is right about it merely serves to demonstrate the strangeness of her theory. Since a

[56] Maddy (1990), p. 97.

[57] Maddy (1990), p. 157, footnote 10.

set-theoretic hierarchy is generated with two objects, but not with one, in her theory, she thinks of the set-operator as forming a new object – {x, y} – out of two objects, but not out of one (since {x}=x). But why should this be? Her motive for identifying {x} with x, when x is a physical object, is simply the fact that there is no perceptible difference between {x} and x. However, this motive is equally pressing in the case of doubleton {x, y}. Supposedly, this object is located where x and y are located. But the difference between this object and the objects x and y is no more perceptible than is the difference between {x} and x. One cannot *see* that the object {x, y} is different from the objects x and y.

On the other hand, there would seem to be a logical reason precluding the identification of a doubleton set with its members. For whereas the doubleton set {x, y} is *one*, the objects x and y are *two*. But how significant is this? We are familiar nowadays with plural quantification: we know that there are sentences in which the subject is ineliminably plural, in that the predicate of the sentence does not attach to the each of the subjects included in the plurality individually (as in e.g. "The men surrounded the city".) Why then should there not be a similarly *plural* identity "x and y are identical to the one set {x, y}" in which the predicate "is identical to the set {x,y}" applies to a plural subject? Admittedly, some have argued that a plural identity in this sense is incoherent, in that it violates the identity of indiscernibles. Suppose Bob and Alice are plurally identical to the one object Xynt. Then Bob and Alice appear to have a property – being two – which Xynt lacks.[58] But perhaps this argument is too swift: Bob and Alice – and hence Xynt – are two people, but one F. So the doctrine of plural

[58] For a variant of the argument that plural identity in this sense is contradictory, see Byeong-Yi (1999).

identity is simply a version of relative identity in the sense advocated by Peter Geach.[59]

On the other hand, relative identity has had a bad press. So the correct attitude in the current state of knowledge is probably scepticism about the logical coherence of the identification of {x,y} with the objects x, y. Accordingly, the operation of set formation *has* to generate a hierarchy from two physical objects, even if it is powerless to generate one from a single physical object. But emphasising in this way that doubleton {x,y} is *one* object, not two as x and y are, and hence distinct from them, makes Maddy's view more puzzling, not less so. For if {x,y} is distinct from x and y, how can it be located where they are? If it is located *in its entirety* where *each* of them is, {x,y} is no longer an object at all in the traditional sense: it is a universal. But if it is located at x only in part, and at y only in part, it would appear that x and y must be parts of it. But that seems wrong too. The doubleton set {x,y} cannot be the object whose only parts are x and y. That would make it indistinguishable from the mereological sum of x and y.[60] And now that we have mentioned it, how is this latter object distinguished from singleton set {{x,y}}? Both are one, and both are located where x and y are. And finally, how is {x,y} to be distinguished from a mixed set like {x,{x,y}}? Both sets are located where the implicated physical objects are located. Hence, both are located where x and y are. But this is to say that infinitely many sets are located there: for {{{x,y}}} is no less distinct from {{x,y}} than is the latter from {x,y}, and so on up through the hierarchy past {{{{x,y}}}} and beyond. That is a lot of imperceptible differences.

[59] Cf. Geach (1967).

[60] The mereological sum of certain entities is the object whose parts are all those entities, together with all of their parts.

The untenability of other versions of 'faint of heart' realism

Maddy's asymmetrical treatment of the singleton sets {x} and {y} on the one hand, and the doubleton set {x,y} on the other, is puzzling. And it may be that on reflection she should relinquish it by distinguishing {x} from x after all, even where x is a physical object. But restoring symmetry in this regard simply exacerbates the problem arising from her insistence that the set-theoretic hierarchy be located in space-time. Whether or not she admits an infinite hierarchy of distinct objects x, {x}, {{x}}, {{{x}}}, ... for each physical object x, she is committed to one of the form {x,y}, {{x,y}}, {{{x,y}}} for each pair of physical objects x,y. We have already noted the incongruity between her contention that sets are perceptible and the obvious fact that at least after the first pair in the hierarchy, the differences between these objects are imperceptible. But the supposition that each of these objects has the same location in space-time – namely, the place where x and y are – is no less incongruous. The conviction that it is impossible for two physical objects to be in the same place at the same time has a long history, and it remains as appealing today as it ever was.[61]

Even were the incongruity of infinitely many imperceptibly different objects being located at the same location put to one side, Maddy's set-theoretic realism encounters a general difficulty, which any version of faint of heart realism will encounter. We might ask if there are enough concrete objects, located in space and time, for classical mathematics in the

[61] It has to be admitted that there has been some movement away from this conviction in recent times, prompted by a desire to distinction between a functional object – such as a statue or ship – and the physical matter of which it is composed (one of the many examples which prompts this distinction is the age-old one of Theseus's ship). But several thinkers have been concerned to restore the conviction by formal devices which accommodate the examples. See especially Lewis (1971).

first place. Are there really *infinite* objects out there? At best, there is an issue about it. As Hilbert points out, even though Euclidean geometry does imply an infinite space, the elliptical geometry offers a model of a finite space and all the physical, that is astronomical, results are compatible with the latter.[62] Einstein's results show that the Euclidean geometry has to be ruled out after all and that a finite universe is possible. This gives more reasons for abandoning the view that mathematical objects, which classical mathematics is dealing with, can be identified with certain concrete, physical objects.

Of course, to suppose that space-time is finite in the large is not to deny that it is infinite in the small: the supposition that space-time is e.g. the surface of a (n-dimensional) sphere, leaves open the possibility of its comprising non-denumerably many points. But infinity of this ilk is nothing to the set-theoretic hierarchy, and hence to mathematics. The cardinality of the continuum barely touches upon the cardinalities in the set-theoretic hierarchy. Are we really to suppose that the cardinality of the physical world reaches up into the remoter regions of the set-theoretic hierarchy? No, surely not. The matter is impossible. The cardinalities of even ZFC are too big. The efforts of theorists such as Maddy notwithstanding, faint of heart realism is untenable. It tries to pack sets in to a space – the physical world – which is simply too small to accommodate them.

2. Platonism

We have now seen that mathematical objects cannot be concrete. Since, as has been previously shown, mathematical objects must exist objectively if the mathematical theorems we

[62] Hilbert's conclusion is that infinity exists just in our thinking. See Hilbert (1925) in Benacerraf and Putnam (1983), p. 186.

hold true are to be true, there is no further option. Platonism itself must be true. That is, mathematical objects are abstract objects. Mathematics studies these objects. Mathematical theorems are true, typically, in virtue of the properties of the abstract objects they concern.

Instead of identifying platonism with a species of realism, as I have, some classifications identify platonism with realism in ontology. However, very few theorists would think of themselves as platonists unless they were prepared to assert realism in truth-value as well as realism in ontology. Partly because of this discrepancy, it is as well to remind us of what platonism in my sense amounts to. Platonism is a version of realism and depicted realism, both about truth value and about ontology. So like other forms of realism, it holds that mathematical statements are objectively true or false, and that, typically, those of them we take to be true are by an large true. In particular, then, the expressions in them that have a referential role are successful in referring. Their truth values depend just on how things are in the mathematical world; mathematical objects which really exist are truth makers for mathematical statement.

Here, 'existence' is not being used in some idiosyncratic or metaphorical sense: realism holds the existence of mathematical objects to be strictly analogous to the existence of physical objects. Sentences like "There are at least three cities older than New York" and "There are at least three perfect numbers greater than 17" have the same logical form, [63] and the notion of existence expressed by the quantifier "there are" which occurs in them is the same.

Thus a statement of the form, 'For some natural number n, A(n)', platonistically interpreted, makes no reference to

[63] See Benacerraf (1973), p. 405.

whether or not we are able to cite some numeral (n) such that A(n) is true, or even to whether we can disprove the statement, 'For all n, not A(n)'. It relates to whether there is a member of the objective domain of natural numbers satisfying the predicate 'A(n)'... independently of our capacity to determine such natural number.[64]

What is specific for *platonist* realism is the claim of *abstractness* - the idea that mathematical objects are, in addition to being mind-independent, also *abstract*, that is non spatio-temporally located. It has to be admitted that this view is not unproblematic in various respects. We therefore need to consider the subtleties of platonism in more depth. This is the task of Part 2.

[64] See Dummett (1978), p. 202.

PART 2

VERSIONS OF PLATONISM IN THE PHILOSOPHY OF MATHEMATICS

5

What Problems Does Platonism Have to Face?

In the previous chapters I have offered reasons for accepting realism. Moreover, in so far as I have rebutted the realistic alternative to platonism - 'faint of heart' realism – I have also offered reasons for endorsing platonism. Some philosophers however think that platonism cannot be true. In the main, their arguments focus both on epistemology and on ontology. But they also discern a tension between platonism's view that mathematical objects are abstract, and the obvious fact that common sense, and, especially, science, successfully applies mathematics to the physical world.

In this chapter I will present these counter-arguments in detail. Different versions of platonism offer different solutions to them. This much will emerge in the following chapters of the second part of the book. But I also offer few solutions that, according to my opinion, could solve the explained problems.

1. The problem of applied mathematics

The 'application' problem arises from the fact that mathematics as a whole, and in particular arithmetic and analysis, are applicable to the physical, empirically perceptible world not just in the sense of being true, but also in the sense of being useful. This problem is hard enough for any philosophy of mathematics to solve. But platonism has seemed to many to make the problem not just hard, but insoluble. Platonists maintain that mathematical objects are abstract, and hence causally inert and without spatio-temporal location. But how could knowledge of objects of this kind possibly be of any use in natural science's attempt to explain and understand the natural, *concrete*, world?

The application problem is a somewhat ironic turning-of-the-tables on the use to which the realist put the fact that mathematics plays an ineliminable role in natural science in his 'indispensability' argument. According to that argument, the role mathematics plays in science provides strong evidence – and perhaps the best evidence we have – that mathematics is true. But while realism is a large component of the doctrine of platonism, it now turns out that the very phenomena to which the platonist might well appeal to substantiate his realism, undermines the nuance which characterises his specific version of realism.

However, in response to this argument I do not think that platonism renders the application problem especially acute. If, to take a crude example, "2+2=4" is true, is it not then obvious that whatever two units - cats or missiles or whatever - we take and add two more of them we will get four of them? The relation between what "2+2=4" is true in virtue of, and what 2Fs plus 2(further)Fs = 4Fs is true in virtue of, can be expressed as follows: 2F+2Fs=4Fs is true in virtue of 2+2 being 4. *That* much is obvious, irrespective of whatever it is in virtue of which 2+2=4. There is simply no question that a platonistic construal of the latter truth makes the truth that 2 concrete F's plus two further concrete F's makes 4 concrete F's. That 2+2=4, whatever feature of the world this fact amounts to, leaves no room for 2F's + 2 more F's to equal 4 F's, whatever F's are in question. Since the sequence 1F, 2F, 3F, ... exemplifies the natural number structure, 2Fs+2Fs=4Fs exemplifies the form "2+2=4". It follows that what makes "2Fs+2Fs=4Fs" true is the fact that "2+2=4" is true. And it does so even if the latter is true in virtue of the properties of independently existing abstract entities.

2. The epistemological problem

The main epistemological problem for platonism was familiar even to Plato. But perhaps its clearest and most prominent formulation in the contemporary philosophy of mathematics is due to Benacerraf[65]. It can be formulated briefly in the following way: if the causal theory of knowledge is true, and mathematical objects are abstract and therefore causally inert, then no mathematical knowledge is possible. The obvious conclusion to be drawn, since we do have some mathematical knowledge, is that platonism is untenable.

Other authors too, as Steiner, Hart or Jubien have formulated the same problem. For example, Steiner writes:

> The objection is that, if mathematical entities really exist, they are unknowable - hence mathematical truths are unknowable. There cannot be a science treating of objects that make no causal impression on daily affairs. All our knowledge arises from the causal interaction of the *objects* of this knowledge with our bodies. Since numbers, *et al.*, are outside all causal claims, outside time and space, they are inscrutable. Thus the - [platonist] - mathematician faces a dilemma: either his axioms are not true (supposing mathematical entities not to exist), or they are unknowable.[66]

The epistemological argument against platonism, as formulated by Benacerraf, is a powerful one. Platonists have to concede that abstract objects are causally inert. Hence, if some causal link with an object of knowledge were a necessary con-

[65] Benacerraf (1973).

[66] Steiner (1975), p. 110.

dition of knowledge, as the causal theory of knowledge supposes, mathematical knowledge platonistically conceived would be impossible. And that much would scotch platonism.

But is the causal theory of knowledge correct? Some formulations of it can clearly be seen to be false, independently of the status of platonism. There can be no question of a must exist a causal chain that connects us with the object(s) of knowledge. However, this chain needn't be immediate. I know that there was a car accident in front of my house last night because I was in causal contact with it: the photons from that scene entered my eyes and allowed me to see the two cars smashed. But in order to know something I do not have to be the first link of the causal chain. I also know that in July 2006 Hezbollah rockets were raining on northern Israel even though I have never been there. I know it because someone was in direct causal contact with the bombarding and recorded it, which led to the recorded tape shown on TV (or to the printed page of a journal) in front of me and then to the photons from the TV screen (or the page in the journal) entering my eyes. I then "saw" the bombardment and came to know that it had happened.

Nor will it do to require a mediated causal link between the object of knowledge and the knower. That would rule out knowledge of the future. John flips the detonator, on which there is an uninterruptible ten second delay. The set up is such that I *know* there will be an explosion. But of course, the explosion is future: it has no causal effect upon me, mediated or otherwise. Causation works forwards in time. The moral the causal theory draws from cases of this sort, of course, is that the causal relationship a knower bears to the object of knowledge can be convoluted, and very indirect. In the straightforward temporal case, I know that the explosion occurs because I know a cause, rather than an effect of it.

However, another sort of case which is more difficult for even the modified causal theory to accommodate is provided by quantum mechanics.[67] Consider an experiment in which a decay process gives rise to two protons moving in opposite directions towards two detectors each situated at one end of a room. Even if the particles are separated by an arbitrarily large distance and can no longer communicate with one another, it is true that: if the z-component of the spin for one particle (let us call it particle 1) is up, then the other particle (particle 2) has spin down, and vice-versa. The result for particle 1 implies the result for particle 2. However, while Einstein, Podolsky and Rosen explained the phenomenon postulating hidden variables the values of which were fixed before the experiment, it was later proved that this postulate leads to results different both from those of quantum mechanics and of experimental results on the spin correlation of the proton pairs.[68] It means that, by reading the results at one detector, we know immediately the result on the other detector without being in *any way* causally connected to the latter one.

Can the causal theory of knowledge accommodate this case? I know of the remote paired particle's spin because I deduce it from my knowledge of the other particle's spin and my knowledge of quantum mechanical laws. Does the latter knowledge establish any kind of convoluted causal relationship between myself and the remote particle's paired spin? Perhaps. It depends in part on the nature of laws, and their relationship with the objects and events which fall under them. Perhaps the quantum mechanical laws are common causes which link my knowledge of them and my knowledge

[67] See Brown (1999), pp. 15-18.

[68] For more details see Schwabl (1990), pp. 383-388.

of the paired particle's spin: they are part of the causal story regarding how that particle came to have the spin it does, and how I came to know them. However, by this point it is clear that the proponent of the epistemological argument against platonism has lost the battle. The dialectic tells against him. Even if he can stretch his causal theory of knowledge to cover the quantum mechanical example, the dialectic has shifted. The platonist started off confronting a *compelling* theory of knowledge, one which rendered mathematical knowledge as conceived by the platonist impossible. But he now confronts a theory of knowledge which its proponent has to admit *may* be refuted by an example which is independent of platonism. In these circumstances the platonist is free to say: even if the quantum mechanical example does not refute your theory of knowledge, the existence of mathematical knowledge – i.e. as I conceive it – does.

Of course, the conclusion I draw here should not be overemphasised. I do not claim to have solved the epistemological problems of platonism. I have not said what mathematical knowledge, as platonistically conceived, consists in, or, even, how it is possible. These are questions to be addressed in later chapters. All I have said is that it is wrong to think that the causal theory of knowledge shows that platonism makes mathematical knowledge impossible.

3. The main ontological problem

In classical mathematics the possibility of reducing any field of mathematics to set theory is a well known result. It means that, in particular, numbers themselves can be identified with certain sets, and that theorems about them can be proved from the axioms of set theory. However, as is well known, it is possible to identify numbers with sets in different ways. Two familiar identifications are with Zermelo's ordinals on the one hand, and von Neumann's ordinals on the other:

Zermelo's reduction	von Neumann's reduction
$0 = \varnothing$	$0 = \varnothing$
$1 = \{\varnothing\}$	$1 = \{\varnothing\}$
$2 = \{\{\varnothing\}\}$	$2 = \{\varnothing, \{\varnothing\}\}$
$3 = \{\{\{\varnothing\}\}\}$	$3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
.	.
.	.
.	.

In mathematics itself these multiple reductions do not cause any problem. Both reductions, as well as all the other possible ones, are equally useful, and there is nothing in the theory of numbers that makes one reduction better than the other.

For the philosophy of mathematics, however, and in particular from the point of view of platonism, these set-theoretic reductions are problematic. Platonism takes the apparent ontological commitments of mathematics at face value, and takes them to be commitments to abstract objects. So it is committed to the existence of numbers on the one hand, and to sets on the other. But if both numbers and sets are objects, *which* objects they are must surely be a determinate matter, no less independent of our mathematical practice than is any other feature of the realm of abstract mathematical objects. The possibility of e.g. identifying the number 2 with two different objects - with the set $\{\{\varnothing\}\}$ as Zermelo does, or with the set $\{\varnothing, \{\varnothing\}\}$ as von Neumann does, can no longer be seen as a healthy richness of resources. It becomes a worrying indeterminacy. *Which* of these sets the number 2 is identical to is a serious question.

This problem for platonism in ontology has also been emphasised by Benacerraf.[69] He calls it the problem of 'indetermi-

[69] Benacerraf (1965), pp. 47-73.

nacy'. It can be summarised like this. Since it is possible to reduce any field of mathematics to set theory, we can reduce numbers too. But there are several possible reductions of arithmetic to set theory. Platonism seems to have the consequence that at most one of these reductions is correct. But how could that be? What *mathematical* facts could make it the case that e.g. Zermelo's reduction is correct, and von Neumann's wrong? And in the absence of mathematical facts which determine this matter, how could the matter be determined? Accordingly, platonism appears to have a false consequence: it appears to have the consequence that certain questions of identity have determinate answers, when in fact they have none.

One might think that platonism has an easy answer to this question: numbers are not *really* sets after all, even if it is convenient for mathematicians to think of them as such. However, this answer is problematic. It implies a certain revisionism with respect to standard mathematical practice,[70] one which is unsatisfactory in itself, and which is counter to most platonist thought. No, the problem of indeterminacy is a serious one. The question remains: since it is possible to reduce arithmetic to set theory in different ways, *which* objects are (natural) numbers?[71] Platonists offer different answers to this question. We will consider them in the remaining chapters of this part.

[70] An anti-revisionist approach in philosophy of mathematics means that classical mathematics has to be accepted and that no philosophical reason can interfere with the mathematical results.

[71] As Benacerraf points out in 'What numbers could not be', Takeuti has shown that Gödel-von Neumann-Bernays set theory is reducible to the theory of ordinal numbers less than the least inaccessible number. Sets are therefore ordinal numbers, but this fact does not influence in any way the above discussion. The question which numbers are which sets (that is which ordinals numbers) still remains.

6

Structuralism

In the previous chapter I have presented the main problems that platonism has to face. In the following chapters I explain and analyse how different versions of platonism offer different solutions to such problems as well as offer my own solutions to them.

In this chapter, my goal is to analyse the solutions offered by platonistic structuralism and try to show that, however appealing structuralism might seem to be, it does not solve successfully the problems it aims to solve.

I firstly characterise different versions of structuralism: in particular, I distinguish those that can be treated as versions of platonism - I will call them platonistic structuralism, and those that can not - I will give this versions the common name of non-platonistic structuralism. Since in the first part of the book I have offered arguments for endorsing platonism, I present briefly the reasons for rejecting non-platonistic structuralism. I then concentrate in more details on platonistic structuralism by examining the ontology of those versions in some depth, and clarifying it. I then describe the ways in which, respectively, platonistic structuralism tries to solve the main problems – of indeterminacy and of epistemology – facing platonism in general and offer my attempted criticism.

1. Versions of structuralism

The basic thesis of structuralism is that mathematics is about structures.

> The mathematics book is not describing a system of sets or Platonic objects or people. It describes a structure or a class of structures.[72]

Intuitively, this idea seems to describe exactly what mathematics is all about. In mathematics we do talk about the group structure, vector space structure and so on, carelessly of the existence of any particular group or vector space. Structuralists, like Shapiro and Resnik[73], claim that the "structuralist" doctrine that mathematics is about structures offers also a solution to the main problems facing platonism: the problem of indeterminacy and the one of epistemology.

Apart from the basic tenet of mathematics being about structures, different versions of structuralism differ in many ways.

According to Dummett, there are two main versions of this doctrine: mystical and hardheaded.[74]

Mystical structuralism comprises two main theses. Firstly, the idea that mathematics is concerned with abstract structures and that the elements of the structures have no properties beside the structural ones, that is, no non-structural properties; mathematical objects are structureless places in structures and are not given in isolation; secondly, (Dedekind's) view that abstract systems are free creations of the human mind. We create systems by psychological abstraction and we need a non-abstract system to begin with. As Dummett rightly points out, Dedekind endorses

> the need to maintain that we can find infinite system of objects - system isomorphic to the

[72] Shapiro (1997), pp.131-2.

[73] See Shapiro (1997); Resnik (1997).

[74] Dummett (1991), pp. 295-296.

> natural numbers and others isomorphic to the real numbers - in nature;...[75]

By contrast, according to the hardheaded version, the elements of a system analysed in mathematics are not mathematical objects since objects cannot have just structural properties.

> It is part of such a view that the elements of the systems with which a mathematical theory is concerned are not themselves mathematical objects, but, in a broad sense, empirical ones; it is not the concern of mathematics whether such systems do or do not exist.[76]

A mathematical theory concerns not just one mathematical system but all systems with a given structure. When we talk about a structure that is just a shorthand for talking about all systems that exemplify the structure. It is a sort of structuralism 'without structures'.

Shapiro[77] introduces a slightly different distinction. Dummett's mystical structuralism is *ante rem* structuralism in Shapiro's terms. *Ante rem* structuralism holds that structures are genuine objects and they exist even if there are no systems of objects that exemplify them.[78] It is the view that mathematics is concerned with abstract structures where a structure is the abstract form of a system and a system is a collection of objects with certain relations.

[75] Dummett (1991), p. 296.

[76] Dummett (1991), p. 296.

[77] Shapiro (1997).

[78] Except for the structure itself. Namely, each structure exemplifies itself since its places, bona fide objects, form a system that exemplifies the structure.

> ...a structure is a pattern, the form of a system. ...Thus, structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token.[79]

A structure is an abstract form of a system in which 'any features [of the objects] that do not affect how they are related to other objects in the system'[80] are ignored. Structures are not necessarily mathematical: we can talk about the natural number structure in the same way in which we talk about a chess configuration. However, even though Shapiro identifies mystical and *ante rem* structuralism they should be distinguished, since Shapiro does not endorse the view that abstract system are creations of the human mind. On the contrary, *ante rem* structuralism can be viewed as a version of platonism and thence I will present it in more details in the next sections. I will also present the platonistic version of structuralism endorsed by Resnik. Even though Resnik mostly agree with Stewart Shapiro, his version of structuralism is different. As a matter of fact, in his previous papers[81] Resnik was endorsing, as Shapiro does, a full-blooded *ante rem* structuralism according to which structures were abstract objects existing independently of any system that exemplified them. In his last book, *Mathematics as a Science of Patterns* that I will concentrate on, he distances himself from such a view.[82] Nevertheless, Resnik has to be included in the talk of structuralist versions of platonism since he is a platonist in respect to mathematical objects and endorses a structuralist view of mathematics.

[79] Shapiro (1997), p. 84.
[80] Shapiro (1997), p. 74.
[81] See, for example Resnik (1975).
[82] Resnik (1997), p. 269.

Shapiro calls Dummett's hardheaded structuralism *'eliminative* structuralism'.[83] He identifies eliminative structuralism with what he calls *in re* structuralism. The *in re approach* holds that there is no structure if there is no system that exemplifies it. The last two versions, even though identical according to Shapiro, have to be distinguished. Namely, even though both versions endorse an *in re* approach to structures, *in re* structuralism does not request that systems consist of empirical objects.

Eliminative structuralism is the version endorsed by Benacerraf. The eliminative part of the label is due to the idea that numbers are nothing more than places in the 'natural number' structure. To be the number 3 means nothing more than to be preceded by 2, 1, 0 and to be followed by 4, 5, and so on; *'any* object can *play the role of* 3'[84], which means that every object can be in the third place i.e. the third element in a progression. Number theory is therefore not about particular objects-the numbers. At first sight this theory might appear as endorsing realism in truth-value and anti-realism in ontology but, as it has been explained in Chapter 3, this is not the case. Benacerraf endorses both realism in ontology (but not an ontology of numbers, but rather of structures) and in truth-value. I will therefore have a closer look at it in the next sections.

There is one more version of structuralism that has to be mentioned. It is the modal eliminative structuralism according to which arithmetic is about all logically possible systems of a certain type. So according to modal eliminative structuralism it is not necessary to assume that a system that exemplifies a given structure exists; it suffices that such a system is logically possible.

[83] Shapiro (1996), p. 150.
[84] Benacerraf (1965), p. 291.

2. Critique of non-platonistic structuralism

In this section I will have a look at versions of non-platonistic structuralism and offer reasons for rejecting them in the light of the arguments for holding platonism offered in the first part of the book.

As far as mystical structuralism is concerned, one of the basic ideas of this version of non-platonistic structuralism is that there exist in nature systems exemplifying the natural number structure or those isomorphic to the reals. That there are infinitely many physical objects out there is, at best, questionable, as I have explained in Chapter 4. The latest physical theories are compatible with the universe being finite.

Apart from this, as Dummett says:

> It may be held, indeed, that time, for instance, has the structure of the continuum; but this seems more a matter of our imposing mathematical structure on nature than of discovering it in nature.[85]

Everything mentioned gives us reasons for refusing such a view.

What about the *in re* structuralism? *In re* structuralism has an *in re* approach to structures: if all the systems that exemplify the natural number structure disappeared the natural-number structure would disappear too.[86] There is no structure unless there is at least one system that exemplifies it. Since this version has an *in re* approach to structures it faces the difficulty of having a robust background ontology in order to make sense of a vast part of mathematics. So, in order

[85] Dummett (1991), p. 296.

[86] It does not seem for the eliminative structuralist to be committed to such a view, nor the other way round.

to keep arithmetic from being vacuous it is necessary to assume that there is a system that exemplifies the structure; that is, that there are enough objects for each structure to be exemplified. And here again the same problem arises, as it was the case for the mystical version.

What about modal eliminative structuralism? Why should we reject it? Modal eliminative structuralism endorses realism in truth-value and anti-realism in ontology. Mathematical statements are true with no ontological commitment to the existence of mathematical objects. This version, even though conceptually coherent, does not give a straightforward account of mathematical language. Namely, in mathematics, numerals - contrary to what the basic idea of modal eliminative structuralism is - seem to function as singular terms; while according to modal eliminative structuralism, mathematics quantifies over systems. Theorems of number theory are here claims about any system that exemplifies the natural number structure (any system with certain properties). But number theory has no variables; it has numerals that - given that mathematical theorems are true - stand for objects. So, there could be a theory as the one proposed by modalists, but it does not seem to be the case with the theories mathematics is actually about (like number theory).

3. The ontology of platonistic structuralism

Two questions arise concerning the ontology of structuralism: one concerning the ontology of mathematical objects, the other concerning the ontology of structures themselves. In this section I will present how these problems are solved according to different versions of platonistic structuralism.

Ante rem structuralism

Ante rem structuralism is an ontological realism about mathematical objects:

> Structuralists hold that a nonalgebric field like arithmetic is about a realm of objects - numbers - that exist independently of the mathematician, and they hold that arithmetic assertions have non vacuous, bivalent, objective truth-values in reference to this domain.[87]

Ante rem structuralism cannot nevertheless be identified with traditional Platonism. According to traditional Platonism it is possible to determine the essence of each number without referring to the other numbers. Accordingly, traditional platonism is non-structuralist.[88]

The problem for the non-structuralist platonist consists in the fact that even though a mathematical theory is about certain entities we cannot definitely determine what kind of objects they are. The non-structuralist platonist view is therefore rejected by structuralists since

> The essence of a natural number is its *relations* to the other numbers. ...
> There is no more to the individual numbers "in themselves" than the relations they bear to each other.[89]

What about the ontological status of structures?

Ante re structuralism

> takes a realist approach, holding that structures exist as legitimate objects in their own right. According to this view a given structure exists independently of any system that exemplifies it'.[90]

[87] Shapiro (1997), p. 72.
[88] Shapiro (1997), pp. 72-3.
[89] Shapiro (1997), pp. 72-3.
[90] Shapiro (1996), p. 149.

According to Shapiro structures are genuine objects. Every structure is a universal and every system that exemplifies it is an instance[91]; the properties of structures are independent of us. 'Mathematical assertions are read at face value, and numerals are singular terms'[92].

There seem to be two difficulties concerning this view:

1) According to Shapiro, numbers are bona fide objects as any objects are, '[M]athematical objects - places in structures - are abstract and causally inert'.[93] On the other hand, he endorses the view that the term 'object' is relative to the theory in question:

> Our conclusion is that in mathematics, at least, one should think of "object" as elliptical for "object of a theory".... The idea of a single universe, divided into objects a priori, is rejected here.[94]

2) Shapiro characterises himself as a 'methodological platonist', with no ontological commitments toward structures. Nevertheless, in his book *Philosophy of Mathematics - Structure and Ontology* he does talk about the objective existence of the natural - number structure:

> The natural-number structure has objective existence and facts about it are not of our making.[95]

That means that the natural-number structure exists independently of us and therefore of our linguistic resources, too.

[91] When Shapiro uses the term 'universal' he refers to a pattern or structure, the 'particular' refers to a system of related objects rather then to an individual object.
[92] See Shapiro's Introduction to the volume four (special issue: Mathematical Structuralism) of the Philosophia Mathematica (3) Vol. 4 (1996), pp. 81-82.
[93] Shapiro (1997), p. 112.
[94] Shapiro (1997), p. 127.
[95] Shapiro (1997), p. 137.

On the other hand, this view seems to make questionable the idea that 'the language characterises or determines a structure (or class of structure) if it characterises anything at all'.[96]

> The point is that the way humans apprehend structures and the way we "divide" the mathematical universe into structures, systems, and objects depends on our linguistic resources.[97]

This view suggests that the distinction between structures and systems does depend of our language and therefore it is difficult to see in which way it is 'not of our making'.

Resnik's version of structuralism

As a structuralist Resnik too endorses the basic features of structuralism, i.e., that mathematics is a science of patterns, where mathematical objects are just positions in patterns[98].

When he talks about patterns he has in mind abstract patterns, which are types, and they have to be distinguished from concrete patterns, which are tokens. To refer to concrete patterns, i.e., tokens, Resnik uses the term 'template'. Templates are usually concrete drawings, models and the like. Apart from these, we can have, and in mathematics they represent the most reliable method for representing patterns, linguistic patterns. These are descriptions of patterns. The example might be the Peano axioms. I will explain the importance of templates later on.

Resnik calls positions in a pattern the one or more objects a pattern consists of, and these positions stands in certain relationship. It is also possible to represent patterns as positions in other patterns. The example might be the number sequence

[96] Shapiro (1997), p. 131.

[97] Shapiro (1997), p. 137.

[98] Resnik uses the terms 'pattern' and 'structure' interchangeably.

that can be represented as a position in a set-theory hierarchy[99]. Resnik compares a position with a point in geometry in the sense that, as a geometrical point, a position cannot be differentiated outside the pattern to which it belongs. Just

> within a structure or pattern, positions may be identified or distinguished, since the structure or pattern containing them provides a context for so doing.[100]

The objects of mathematics are just structureless positions in patterns and they have no identity outside a pattern, no 'identifying features independently of a pattern'[101]. Talking about mathematical objects is a way of talking about patterns and their positions.

What is a pattern? Resnik does not define a pattern via its instances because in that case there would be necessary to require a prior ontology of instances. 'Otherwise, all universal quantifications concerning uninstatiated patterns would be vacuously true'.[102] And since there is not enough physical objects to guarantee all mathematical patterns to be instantiated there would be necessary to posit mathematical objects.

On the other hand, positing mathematical objects that are not taken as positions in a pattern would mean rejecting the main structuralist's idea. So, in mathematics, we posit mathematical objects as positions in a structure when there are not enough mathematical objects among those we already have in order to get the structure we want to model.[103] Mathemat-

[99] Resnik (1997), p. 240.

[100] Resnik (1997), p. 203.

[101] Resnik (1997), p. 211.

[102] Resnik (1997), p. 204.

[103] The example Resnik mentions is the introduction of the complex numbers to obtain an algebraic closure of the reals.

ics also produces new patterns by adding new positions to the patterns it already recognises. Science experience as well suggests new patterns, generating new branches of mathematics.[104]

Let us have a look now at the ontology Resnik's endorses.

As it was the case in the section about Shapiro's version of structuralism, there are two questions that need to be answered: one is the question about the ontological status of patterns and the second one of instances.[105]

As it has been mentioned in parts three and four, according to Resnik mathematical objects are positions in patterns and they are abstract, non spatio-temporal located. The ontological reading of his version of structuralism that he has in mind is the one that treats structuralism

> as positing an ontology of featureless objects, called 'positions', and construing structures as systems of relations or 'patterns' in which these positions figure. Under this construal, structures are like relations in extension whose *relata* are positions.[106]

Even though, according to Resnik, mathematics studies structures and mathematical objects are positions in structures, he avoids treating patterns as entities[107] of any kind.

Resnik does existentially quantify over patterns since a mathematical theory does require individual variables rang-

[104] See Resnik (1997), p. 242.

[105] Respectively structures and systems in Shapiro's terminology.

[106] Resnik (1997), p. 269.

[107] Resnik uses the term 'entity' as a generic term for anything of any logical type, while uses the term 'object' and 'individual' for things of the lowest logical type (first-order entities).

ing over patterns but as a sort of shorthand for certain mathematical statements, not to be understood at face value.

Structures the typical mathematical theory describes do not belong to its universe of discourse, current mathematics does not affirm their existence and not recognise them as mathematical objects[108]; that is why realism about mathematical objects is not committed to realism about structures. The idea is therefore that a position in a structure can be real even though structures are not.

> Treating structures as individuals would undo these parallels with mathematics, since it would permit identities between patterns, and this in turn would permit identities between their positions.[109]

Apart from that, 'there is no fact of the matter as to whether the patterns various mathematical theories describe are themselves mathematical objects (positions in patterns)'[110] since some mathematical theories identify structures with mathematical objects while others do not.

On the other hand, theorems of a certain part of mathematics are true of the structure they describe and this claim contains two assertions:

1. the assertion that theorems logically 'follow from the clauses defining the structure in question', i.e., are consequences of the definition of the structure in question, and

[108] For example, set theory quantifies over sets but not over the set-theoretic hierarchy.

[109] Resnik (1997), p. 211.

[110] Resnik (1997), p. 243.

2. the existential claim, i.e., the assertion that 'there are structures (or at least appropriate related positions) of the kind in question'.[111]

There are several unclearities that arise in such a version of structuralism and I will try to point them out:

1. Resnik gives two reasons for not treating structures as entities of any kind:

a) the typical mathematical theory does not affirm their existence, does not quantify over structures

b) some mathematical theories identify structures with mathematical objects while other do not, wherefore, 'there is no fact of the matter as to whether structures are objects of any kind'.

It seems that by accepting the first argument the second becomes superfluous. If the fact that the typical mathematical theory (whatever that means) does not quantify over structures is a good reason for not treating structures as (mathematical) objects then it is irrelevant what approach other mathematical theories might have. If it is not irrelevant then the first argument cannot be a good argument on its own.

But, what is a typical mathematical theory? If such a theory is the one that can be found in every standard book of higher mathematics, then the argument a) is simply false.

The example might be the set theory. In set theory the axiom of infinity precisely affirms the existence of the natural number sequence which, explicitly or not, means the assertion of the natural number structure. Namely, there are two possible points of view:

i) the natural number structure is the natural number sequence

[111] Resnik (1997), p. 240.

ii) the natural number structure is exemplified by the natural number sequence.

In both cases the existence of the latter implies the existence of the former: in the first option it is obviously true, in the second it follows from a structuralist point of view, independently of the various versions.

2. Resnik anyway cannot affirm the first option because that would mean either treating patterns as entities of some sort which is in collision with his main point about patterns[112] or denying the natural number sequence to be some sort of entity which would make problematic the assertion that numbers are mathematical objects.[113]

Therefore Resnik seems to be forced to endorse the view ii), i.e. that the natural number sequence exemplifies the natural number structure.

But that makes problematic the example Resnik gives while talking about the possibility for patterns to be positions in other pattern. Namely Resnik's example is the natural number sequence that can be represented as a position in a set-theory hierarchy. It follows that the natural number sequence after all *is* identified with the natural number structure. So neither option, i) or ii) above, seems to be plausible with other Resnik's views.

3. Resnik's example of patterns being positions in other pattern produces further unclearity: patterns should be mathematical objects since positions are mathematical objects. If patterns are not entities of any kind than it should not be possible for them to be positions in other patterns.

4. The next puzzling point is about structures themselves. A structure is composed by its positions where positions are

[112] That is, his view that patterns are not treated as entities of any kind.

[113] This is one of the main theses of his version of structuralism too.

mathematical objects. Is it a coherent view to say that something composed by objects could not be itself an entity of any kind? Or could not be an object? Apart from that, linguistic patterns describe patterns: that means they do describe something. It seems necessary for structures then to be entities of some kind. What could linguistic patterns describe if structures were not entities of some kind?

5. The existential claim (see above) is in contradiction with the view that there is no fact of the matter as to whether structures are entities of any kind. If a theorem being true of a structure implies the existential claim that there are structures than it cannot be just a shorthand not to be understood at face value. If, on the other hand, all those theorems have not to be understood at face value than Resnik has no reason at all to be doubtful in denying structures to be entities of any kind.

6. If the typical mathematical theory does not affirm the existence of structures than it is not clear to which theory the theorems Resnik is talking about in the existential claim, are related to.

7. According to Resnik's version of structuralism, we introduce mathematical objects by positing them as structureless positions in (abstract) patterns. But, while those positions are objects, patterns are not treated as entities of any kind. Both patterns and mathematical objects are posited though. Mathematical do not treat structures as entities of any kind in the same way in which it does not treat, for example, numbers as abstract objects or structureless positions in patterns. There seems to be no more reasons for treating mathematical positions in structures as existing abstract objects than it is for treating structures as entities of some kind.

4. Structuralism and the problem of indeterminacy

The problem of indeterminacy, formulated by Benacerraf[114], has been explained in details in Chapter 5 (Section 3). *Ante rem* structuralism and Resnik's version of structuralism try to solve the problem in indifferent ways. I will nevertheless start explaining the way in which eliminative structuralism, endorsed by Benacerraf, try to solve the problem since it was him who first pointed it out.

Eliminative structuralism - the solution to the problem of indeterminacy

In his article 'What numbers could not be' Benacerraf tries to solve the problem of indeterminacy, that of identification of numbers with some sort of sets by saying that, since to identify the numbers with sets there are different possibilities, numbers cannot be sets after all.

We might, for example, identify numbers both with Zermelo's and von Neumann's ordinals (there are as a matter of fact infinitely many possibilities). The problem is acute because there appear to be no argument for settling it; there is no way to determine the truth value of sentences like the identity '$2=\{\emptyset,\{\emptyset\}\}$'. Benacerraf concludes that there is 'no "correct" account that discriminates among all the accounts satisfying the conditions ..'.[115]; the only possible conclusion is therefore that numbers could not be sets at all.

Benacerraf extends the argument for the assertion that numbers can't be sets to the conclusion that numbers are not objects at all. The problem of trying to identify numbers with

[114] See Benacerraf (1965), pp. 47-73.

[115] Benacerraf (1965), p. 281.

some sort of objects is, according to Benacerraf, simply pointless:

> The pointlessness of trying to determine which objects the numbers are thus derives directly from the pointlessness of asking the question of any individual number.
>
> ...
>
> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* - and the distinction lies in the fact that the "element" of the structure have no properties other than those relating them to other "elements" of the same structure.[116]

The solution that Benacerraf offers is the view that mathematics is about structures. He adopts 'eliminative' structuralism: numbers are nothing more than places in the 'natural number' structure. To be the number 3 means nothing more than to be preceded by 2, 1, 0 and to be followed by 4, 5, and so on; '*any* object can *play the role of* 3'[117], which means that every object can be in the third place i.e. the third element in a progression. Number theory is therefore not about particular objects-the numbers; it is about the properties of all the systems of the order type of the numbers. Benacerraf is, as a matter of fact, denying the existence of numbers, i.e. he identifies numbers with numerals:

[116] Benacerraf (1965), p. 291.

[117] Benacerraf (1965), p. 291.

> there are not two kinds of things, numbers and numbers words, but just one, the words themselves.[118]
>
> . . .
>
> in counting, we do not correlate sets with initial segments of the numbers as extra linguistic entities, but correlate sets with initial segments of the sequence of number *words*.[119]

In this way Benacerraf avoids the question of what kind of objects numbers are and therefore questions like the Frege's "Caesar problem": 'whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or is not'.[120]

But once we accept, as Benacerraf holds, that '*any* object can *play the role of* 3' and therefore 'any system of objects, sets or not, that forms a recursive progression must be adequate'[121], a problem nevertheless arise.

Let us take the example in which we want to apply numbers to the real world; we need them to determine for example how many letters the word 'hand' has (or how many hands we have). Now, if the only thing that matters is just the structure and if 'any system of objects, whether sets or not, that forms a recursive progression must be adequate'[122] then it should be the same which progression we choose in order to count, i.e. which model of the natural number structure we use. But, we would rather have *four* letters in the word 'hand' (as well as *two* hands) which means that the progression we are looking for is *a specific* one, the one which allows us to

[118] Benacerraf (1965), p. 292.

[119] Benacerraf (1965), p. 292.

[120] Frege (1884), § 56.

[121] Benacerraf (1965), p. 290.

[122] Benacerraf (1965), p. 290.

have four letters (and exactly two hands) and in which, if we add one more letter, the numbers of letter will be five. There are infinitely many progression unacceptable in that sense: let us take the progression 0, 2, 4, ... which has the natural number structure; if we accepted this one we would have 4 hands and the word 'hand' would have 8 letters. We would say that such a progression is inadequate even though the progression is perfectly acceptable in pure mathematics and it exemplifies the natural number structure. If this is so, that means that in the application we ask for certain conditions that are not just structural, therefore not every progression is adequate, on the contrary, just one seems to be suitable.

How does *ante rem* structuralism solve the problem of indeterminacy?

As it has already been explained, *ante re* structuralism is the doctrine according to which mathematics is concerned with abstract structures[123] and the elements of the structures have no properties beside the structural ones i.e. no non-structural properties. Mathematical objects (numbers, sets, etc.) are just places within structures; e.g. real analysis is about the real number structure and everything we can say about real numbers consists in their 'structural' properties. It is not possible to postulate one real number because that would mean postulating one place within a structure which is not possible without invoking the structure as a whole. Mathematical entities have no internal properties and they are just positions in structures. It follows that they do not have identity outside the structure either;

> the various results of mathematics which seem to show that mathematical objects such as numbers do have internal structures, e.g., their identification with sets, are in fact interstructural relationships.[124]

[123] To say that a structure is abstract is to say that it can have more then one exemplification.

According to Shapiro, even for a realist in ontology, questions like the Caesar problem need not to be answered, i.e., there is no answer:

> ...it makes no sense to pursue the identity between a place in the natural-number structure and some other object,... Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures.[125]

We have to ask question that are internal to the natural-number structure if we want to get determinate answers because mathematical objects are tied to the structure whose places they occupy. So, even though, differently from Benacerraf, *ante rem* structuralists endorse the view according to which numbers *are* objects, they are objects of arithmetic. We can therefore ask questions about numbers if such questions are internal to the natural number structure, i.e. if they are about relations that can be defined in the language of arithmetic.

Adopting structuralism i.e.

> viewing mathematical objects as positions in patterns leads to a reconception of mathematical objects which defuses the objection to platonism based upon our inability to completely fix their identity.[126]

[124] Resnik (1981), p. 530.
[125] Shapiro (1997), p. 79.
[126] Resnik (1981), p. 530.

I find there are two main difficulties in Shapiro's theory:

1) According to *ante rem* structuralism, the places in the natural-number structure can be occupied by places in other structures. Let us suppose, for the sake of argument, that the objects b_1, b_2, ..., b_n, places in the S_1 structure (there might be infinitely many objects), either exemplify the structure S_2 or occupy the places in such a structure. So the element b_1, ..., b_n have either certain properties p_1,..., p_k which make them exemplify the structure S_2 or properties which are internal to the structure S_2 and which do not correspond to the properties q_1, q_2, ..., q_m which are internal in respect to the structure S_1. In that case the properties p_1,...,p_k are external, i.e. non structural in respect to the structure S_1 and the properties q_1,...,q_m are external in respect to S_2. It is not clear then in what way the objects b_1, b_2, ..., b_n have no non structural properties in relations both to the structure S_1 and S_2.

2) It might also be difficult to say in which way some properties of real numbers such as being transcendental can be treated as structural; this property appeals to the notion of polinomial which seems to be external to the structure.

The problem of indeterminacy and Resnik's solution to it

According to Resnik, there is no fact of the matter, for example, as to whether numbers are sets. One reason to endorse this claim is that mathematics neither denies nor affirms it. Resnik also reckons that the sentence 'numbers are sets' might not even be contained in the mathematical part of our language. But since it obviously can be constituent of the language, in saying that there is no fact of the matter as to whether certain mathematical objects are identical to others Resnik is actually denying truth-values to this and related sentences.

More than this; Resnik namely says:

> I am committed to more than denying truth-values to whole sentences, such as 'each number is a set'. For I have also held that for any number (or more generally, any thing) there may be no fact of the matter as to whether it is a set, i.e. whether 'set' is true of it.[127]

In order to solve the problem of incompleteness of mathematical objects Resnik restricts bivalence, i.e. the universal applicability of classical logic. More precisely, he restricts logic so that the law of excluded middle does not apply in general.

Apart from these reasons, there is no evidence, either in mathematics or science in general, that counts pro or against numbers being sets. It looks as if, Resnik says, the only one interested in having the answer are philosophers.

> Thus letting excluded middle lapse in these cases seems to have the benefit of resulting in a simpler philosophical account of mathematics without generating untoward reverberations elsewhere.[128]

Finally, Resnik assertion that concerns the lack of identity between mathematical objects is a consequence of his philosophical premises; it derives from his thesis concerning the nature of mathematical objects, that is, their being positions in patterns.

What are the difficulties with such a view?

The first reason Resnik gives for not treating numbers as sets seem to be problematic. Namely, the main reason is that mathematics itself does not either deny or affirm so. Why is it not a good reason? Because mathematics is neutral to the point, it is not interested in giving such an answer simply be-

[127] Resnik (1997), p. 245.

[128] Resnik (1997), p. 246.

cause it is not its domain of interest. In books of higher mathematics quite often several possible identification of numbers are explained and it is of no interest for mathematics to question the problem of indeterminacy. Mathematics dos not concern itself with such a problem; actually, it does not see it as a difficulty because it does not enter in philosophical discussions about the existence of numbers. It simply says nothing about it. A mathematician who follows Field's nominalism in philosophy of mathematics and a hard core platonist can develop exactly the same mathematical theory and unless they do not explicitly talk about the existence of mathematical objects they will never find out from their mathematical results the philosophical position of each other. To say that there is no fact of the matter as to whether numbers are sets because mathematics neither denies nor affirms is like saying that there is no fact of the matter as to whether the sun is moving around the earth or the other way round since meteorology says nothing about it.

5. Structuralism and the epistemological problem

Structuralism allegedly solves the epistemological problem for platonism, too. In fact,

> If we conceive of the numbers, say, as objects each one of which can be given to us in isolation from the others as we think of, say, chairs or automobiles, then it is difficult to avoid conceiving of knowledge of a number as dependent upon some sort of interaction between us and that number.
>
> ...
>
> I also think that viewing mathematics as a science of patterns promises to solve the platonist's epistemological problems as well-or at least to make them less urgent-by showing that mathematical knowledge has a fairly central place in our general epistemological picture.[129]

What is the structuralist's epistemology about?

The epistemology in Shapiro's *ante rem* structuralism

According to Shapiro, there are three ways of grasping a structure: abstraction or pattern recognition, linguistic abstraction and implicit definition.

One way of grasping a structure is through abstraction (or pattern recognition). We abstract a structure from one or more systems that have the same structure and grasp the common relations among the objects. This way is analogous to the way in which we grasp the type of a letter by observing different tokens of the letter and ignoring what is specific to a singular token like the colour, the height and the like. By abstraction we grasp small cardinal structures (the first few finite cardinal or ordinal structures) and it works the same as in the case of characters and strings: the child learns to recognise the 4 pattern after different groups of 4 objects have been pointed out to them. The next problem is how to grasp large cardinal structures (and then infinite systems and structures, too). Large cardinal structures are not apprehended by simple abstraction but children learn, during their linguistic development, to parse tokens of strings they have never seen and strings that may have no tokens at all:

> At some point, still early in our child's education, she develops an ability to understand cardinal and ordinal structures beyond those that she can recognize all at once via pattern recog-

[129] Resnik (1981), p. 529.

nition and beyond those that she has actually counted, or ever could count.[130]

In order to grasp the natural number structure we have to reflect on sequences of strokes that becomes longer and longer and form the notion of a never ending (in one direction) sequence of strokes:

> This is an infinite string, and so I cannot give a token of it in this book. The practice is to write something like this instead: | | | | . . . The point is that students eventually come to understand what is meant by the ellipses ". . . "[131]

To obtain structures larger then the denumerable ones, we have to contemplate sets of rationals (as in Dedekind cuts) and in this way we contemplate the structure of the real numbers; we are talking in this case about *linguistic abstraction*. The third way to grasp a structure is through a direct description of it i.e. through its *implicit definition*, e.g., we can grasp the natural number structure by understanding the Peano axioms which are its implicit definition. Shapiro defines the implicit definition in the following way:

> In the present context, an implicit definition is a *simultaneous* characterization of a number of items in terms of their relationships *to each other.* In contemporary philosophy, such definitions are sometimes calls 'functional definitions'[132]

Both implicit definition and deduction support the view that mathematical knowledge is *a priori* :

[130] Shapiro (1997), p. 117.
[131] Shapiro (1997), p. 119.
[132] Shapiro (1997), p. 130.

> Thus, if sensory experience is not involved in the ability to understand an implicit definition, nor in the justification that an implicit definition is successful, nor in our grasp of logical consequence, then the knowledge about the defined structure(s) obtained by deduction from implicit definition is a priori.[133]

So, according to structuralists, structuralism resolves both

a) 'the plight of the mathematical Platonist arising from the existence of multiple reductions of the major mathematical theories'[134] and

b) the epistemological problem for platonism due to the causally inert abstract mathematical entities)

It is time now to have a look at what I consider to be the main problems for Shapiro's answer to the epistemological problem.

1) How can we grasp a structure?

According to Shapiro, since structures are abstract, we do not have any causal contact with them. We do grasp small, finite structures by abstraction via pattern recognition.

> A subject views or hears one or more structured systems and comes to grasp the structure of those systems....The idea is that we grasp some structures through their systems just as we grasp character types though their tokens.[135]

This is how children grasp different types, e.g., letters: by looking at different tokens of letters showed them by their parents representing the same type. But, are types not prior

[133] Shapiro (1997), p. 132.
[134] Resnik (1982), p.95.
[135] Shapiro (1997), p. 11.

to tokens? Do we not have tokens in order to represent types, rather then the other way round? Children can learn about types through tokens because tokens have been already 'assigned' to types; the way we learn and the way we grasp are not necessarily the same.

2) What about infinite structures?

The way we grasp the natural-number structure is through its implicit definition, i.e. through a direct description of it. That means that we are supposed to grasp the natural-number structure via the understanding of the Peano axioms even though Shapiro does not say what it means to understand the Peano axioms. It seems again a way of learning about the natural-number structure rather then a way of grasping it. Are the Peano axioms not the description of the natural-number structure we have somehow already grasped and we want to describe? If the Peano axioms are a description of the natural-number structure, how can we describe a picture before grasping it in the first place? Does that not beg the question? Shapiro's describes the implicit definition as a 'common and powerful technique of modern mathematics':

> ...Typically, the theorist gives a collection of axioms and states that the theory is about any system of objects that satisfies the axioms. As I would put it, the axioms characterize a structure or a class of structure, if they characterize anything at all.[136]

Here again, it is unclear how does the theorist get the (Peano) axioms? Are they a result of the theorist's imagination or does he grasp them in some way? If they are a result of the theorist imagination then it is unclear how we know that a structure corresponds to them; if the theorist grasps

[136] Shapiro (1997), pp. 12-3.

them, the question is 'How?'. He could not have grasped them by grasping the structure since structures are abstract and causally inert (structuralism is supposed to solve the problematic platonistic epistemology). He could have grasped them by grasping a system which exemplify the natural-number structure: one possible answer is by grasping the numerals which Shapiro denies;[137]the other possible answer is by grasping a spatio-temporal system which exemplify the natural-number structure. What about the real-number structure or other infinite structure? Since Shapiro is reluctant to assert the existence of an enough big number of physical objects in the universe when he criticises eliminative structuralism he concludes:

> Because there are probably not enough *physical* objects to keep some theories from being vacuous, the eliminative structuralism must assume there is a large realm of abstract objects. Thus, eliminative structuralism looks a lot like traditional Platonism.[138]

According to Shapiro, one of the reasons why *ante rem* structuralism is the most acceptable version of structuralism is because it does not require a strong background ontology to fill the places of various structures. But it seems that Shapiro might be after all committed to the existence of a 'large realm of abstract objects' as well as the eliminative version.

3) None of Shapiro's suggested ways of grasping a structure explains how it is possible, if it is possible at all, to grasp a structure which no systems exemplify it, except for the structure itself. It seems that grasping such a structure would

[137] As Shapiro points out: 'I do not claim that the natural-number structure is somehow grasped by abstraction from numerals'. See Shapiro (1997), p. 137.

[138] Shapiro (1997), p. 10.

be as problematic as grasping mathematical objects for platonism is; for the reason that structures are abstract and causally inert. The other versions of structuralism do not have to deal with such a case since, according to them, there are no structures with no system that exemplifies them.

Resnik's answer to the epistemological problem

Given that according to Resnik mathematical objects exist in the way that has already been explained, the problem is how do we come to know anything about them?

> Although mathematical posits exist independently of our postulating them, they are not independently given to us. We must introduce terms for them and posit them in order to recognize them.[139]

> The causal theory seems to be so prevalent among realists because we tend to think of theory construction as a matter of first discovering, then naming, and finally describing reality. But often in positing we describe first, and only later obtain evidence of the independent existence of our posit.[140]

So according to Resnik, we come to know about mathematical objects by positing them where

> positing mathematical objects involves nothing more mysterious than the ability to write novels, invent myths, or theorize about unobservable influences on the observable. For to posit mathematical objects is simply to introduce discourses about them and to affirm their existence.[141]

[139] Resnik (1997), p. 188.

[140] Resnik (1997), p. 192.

This 'postulational epistemology', as Resnik calls it, unable us to know about mathematical objects without having to involve any mystical grasping of abstract objects; all we need are our ordinary faculties. Nevertheless, positing mathematical objects differs from positing, e.g. ghosts in the sense that positing the former leads us to knowledge while positing the latter does not. Simply because, unlike ghosts, novels' characters and the like, mathematical objects exist and the rationale for positing new mathematical objects is different from that for creating new fictional characters: we posit new mathematical objects when we find a necessary consequence from our previous results and look for their evidence.

Resnik gives reasons for accepting realism and if mathematical objects exist they have to be abstract. And positing them is, according to Resnik, what mathematicians have been doing through centuries.

Mathematical posits answer to a need such as extending a domain of previous results or answering still undecided questions. They also must fulfil standards of clarity, rigour, and coherence that it is not the case with fictional characters.

> As I emphasized, positing numbers and sets no more calls into question their independent existence than positing the planet Neptune or quarks does theirs.[142]

Finally, unlike fiction, mathematics is indispensable to science, so we need to presuppose its truth when we investigate the world.

[141] Resnik (1997), pp. 184-5.

[142] Resnik (1997), p. 272.

The goal of positing mathematical objects is to describe patterns but I will postpone talking about it for the moment.

There are two more questions that need to be answered though: the Genesis question and the Criterial question.[143] The former asking *how* we come to use a certain term to refer to a given object and the latter asking *when* a given term refers to a given object. In answering them Resnik has to rule out the causal theory of reference since, according to him, there is no causal relationship between us and mathematical objects. His answer for the criterial question is the immanent theory of reference. It is the theory that, unlike the transcendent one, applies only to our own language. The general form of the immanent answer to the criterial question is:

a) for singular term: For any x, the singular term 't' refers to x iff x=t, where t is a schematic letter standing for (in this case) English singular terms.

b) for predicates: For any x, the predicate 'F' refers to x iff x is F, where F is a schematic letter standing for one-place (in this case) English predicates. So,

> Using this theory of reference, there is no special problem with referring to mathematical objects. The predicate 'number', for instance, refers to an object if and only if it is a number. End of story. ... in using 'number' to refer, we refer to something existing independently of our constructions, proofs, and so on, since our constructing a mathematical object or proving theorems about it is not necessary for its existence.[144]

The answer to the Genesis question is given by analysing the history of our use of terms, that is the way in which mathematicians (from the ancients until now) have come to use terms in order to refer to objects. First of all, we do know that our terms refer since mathematical objects exist and this point is crucial, together with the immanent approach to reference:

[143] Resnik (1997), p. 191.
[144] Resnik (1997), p. 193.

> for without them it would seem an explicable coincidence that our mathematical terms happen to refer to a mathematical reality that exist independently of our positing it.[145]

Apart from giving an answer to the question of grasping mathematical objects in general, Resnik analyses also the way we grasp a pattern.

As it has been already mentioned in part three, the point of positing mathematical objects is to describe patterns. In that case, Resnik says,

> it is plausible to allow that systems of physical objects instantiating these patterns can inform us of properties of mathematical objects.[146]

And this is what, according to Resnik, the ancients actually did. The human knowledge of patterns began with experience, that is with templates. One example might have been the dot system to count or record counts. By using the dot system we (or the ancients) could conclude that addition and multiplication are associative and commutative, that multiplication distributes over addition, and that 1 is the multiplicative identity. It was also possible to obtain results about numbers that could evolve eventually in a number theory:

> For example, to add two numbers, one simply dot-adds the dot patterns the numbers now represent and assigns the number representing

[145] Resnik (1997), p. 195.
[146] Resnik (1997), p. 224.

> the resulting dot pattern as the sum of the two numbers.[147]

When talking about pattern cognition Resnik emphases two points. Firstly, by making a distinction between knowledge of a pattern and pattern recognition: to recognise a pattern means just to distinguish those systems that instantiate it from those that do not. So, it is possible to be able to recognise a pattern without having the capacity to describe it. Secondly, by saying that seeing or intuiting a pattern should not be understood at face value. After all, there is nothing to see in a pattern apart from its positions. To see or intuit a pattern means to realise that certain system instanties it, i.e., 'that certain of its instances fit it or satisfy its defining conditions'.[148]

What can lead as to knowledge of (abstract) patterns is the experience with templates and it is also how we (or the ancients) might have begun the exploration of patterns.

> For although initially templates represent only patterned, concrete things, they eventually also come to represent the abstract patterns that concrete things might fit.[149]

In mathematics too, by working with templates and analysing them it is possible to get results from which we could develop a theory of numbers. But, since it is not possible to have infinite templates, it is necessary to turn to linguistic template, i.e., to use axioms eventually.

By taking mathematical objects as positions in patterns it is possible therefore to explain the incompleteness of mathematical objects, mathematics' methodology and epistemology

[147] Resnik (1997), p. 232.
[148] Resnik (1997), p. 225.
[149] Resnik (1997), p. 228.

as well as why mathematical objects do not play a causal role in kenning mathematics.[150]

There are few unclearities in Resnik's views concerning epistemology though:

1) There are facts in the history of mathematics that make positing mathematical objects as Resnik describes it questionable. The example might be the introduction of complex numbers. Complex numbers were introduced by Cardano in his work *Ars magna* in 1545. The introduction was made *ad hoc*, in order to have a more harmonious theory of equations of third order and their solutions. For 300 years mathematicians did not have an interpretation of those numbers[151]and had no idea what those numbers could either mean or represent. No standard either of clarity or rigour was fulfilled: Cardano was treating 'imaginary' numbers as if they were real ones. They were not the necessary consequence of previous results and did not seem those 'imaginary' numbers were indispensable for science at all. It really seemed like positing a new character in a good crime story to make it more exciting and harmonious.

2) According to Resnik, the point of positing mathematical objects is to describe patterns and physical objects that instantiate these patterns can inform us of properties of mathematical objects. Is it really so?

Mathematical objects as positions in a pattern and physical objects instantiating the same pattern have in common those properties that make them having the same structure. We come to know about mathematical objects by positing them as positions in the pattern and we do find out, supposedly empirically, that certain physical objects exemplify the (same)

[150] Resnik (1997), p. 272.
[151] That is the reason why complex numbers were called 'imaginary' numbers for years; they were thought to be the result of human imagination until Gauss in 1831 gave a geometric interpretation of complex numbers as ordered pairs of real numbers.

structure. Mathematical objects are causally inert, non spatio-temporal located and undetectable. Therefore, apart from instantiating the same structure, mathematical and physical objects have nothing in common. It is not hence clear that it could be possible to find out about mathematical objects from physical ones. Besides, according to Resnik, mathematical objects are structureless positions in patterns so there is, as a matter of fact, nothing more to know about them once we know their being positions in a certain structure.

7

Frege's Logicism (1) - Background

In a book on platonism in the philosophy of mathematics a consideration of Frege's view is certainly obligatory. His version of platonism constitutes a refined and monumental development of the doctrine to which all contemporary philosophies of mathematics are at least in part a reaction. In this chapter and the following two, I articulate as much of Frege's theory as is necessary for an appreciation not only of its power but, ultimately, of its failings too.

I begin by describing, in this chapter, the overall framework in which Frege's philosophy of mathematics is developed.

1. Methodological background (1) – Context

Let us start with few historical remarks. That Frege was born in the same year in which Bernard Bolzano died has a symbolic significance. Bolzano's work represented a new approach to mathematics and its foundations. He insisted upon the necessity of eliminating every hint of intuition, of ruling out geometrical diagrams, and of introducing rigour in mathematical proofs even for theorems that appeared to be self-evident or intuitively obvious. For many mathematicians at the time, this seemed to be an unnecessarily severe condition. But Bolzano was concerned to prove theorems that seemed obviously true, such as the theorem asserting that for every function f, continuous on the closed interval [a, b] such that $f(a)f(b) < 0$, there exists a value $c \in (a, b)$, such that $f(c) = 0$. Geometrically, this means that, for every uninterrupted line (that is, the graph of a continuous function) running from below the x-axis to above it (or the other way round), the line must cross the x-axis at some point. Of course, how could such a line possibly not cross the x-axis? Why should we bother to prove such theorems rigorously when their truth is obvious?

In part, we should do so because a statement can seem to be intuitively clear and obviously true and still be false. Intuition can be misleading.[152] But although he endorses Bolzano's project, Frege gives it a somewhat different rationale. Likewise attempting to dispel intuition and visual representation from mathematics, Frege goes one step further. His concern with the foundations of mathematics is not just with the *justification* of mathematical theorems. It is with *rational order* by which such justifications should proceed. As Frege declares:

> after we have convinced ourselves that a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely?[153]

In effect, Frege's idea is not merely that mathematicians should be rigorous in their search for subjective certainty. They should also concern themselves with the *objective* structure of mathematical knowledge.

Of course, the objective "foundationalism" exhibited in this approach to the philosophy of mathematics is not novel. Descartes exhibited it two and a half centuries earlier in his *Meditations'* preoccupation with the shift from a subjective certainty to an objective certainty in keeping with the true "order" of knowledge. However, the way in which Frege implements the demand for objective foundations is strikingly original. His fundamental idea is that the theorems of arithmetic (i.e. elementary number theory) are "analytic". A distinction between statements or propositions which are analytic, and those which are synthetic, goes back to Kant. However, Frege has a rather unusual analytic-synthetic distinction in mind. For him, the distinction turns on the ultimate ground on which the justification for holding a statement true is based.[154] He says:

[152] Although mathematician's judgements as to what is "obviously true" can be mistaken, the possibility of their being so should not be overemphasised. See above, Chapter 2.

[153] Frege (1884), § 2.

> When ... a proposition is called a priori or analytic in my sense, ... it is a judgment about the ultimate ground upon which rests the justification for holding it to be true ... The problem becomes ... that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one... If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the

[154] Dummett (Dummett (1991),pp. 28-29), as well as other authors (see e.g. Shapiro (2000), p. 108); Kitcher (1979)), finds analycity to be for Frege an *epistemic* concept, turning on how a given proposition is known (or knowable) but I find it controversial. It is not clear the distinction Frege draws has anything to do with the way we actually grasp or could grasp mathematical terms. 'The ultimate ground' is, given the rest of Frege's theory, objective, therefore independent of our capacity of grasping it. Frege explicitly says (Frege (1884), § 3) it has nothing to do with the way we come to know a proposition. It could be said that the concept of the proof or justification of a proposition is, as such, epistemic but, according to platonism, a proof exist independently of us and out capacity to now it. In this sense to say that mathematical statements are provable in terms of logic alone (just) means that such a proof contains exclusively logical terms and definitions. Frege does talk of *our finding* the proof but he thinks that all mathematical statements and proofs are knowable so there is extensionally no distinction between what we (can) know and what there (objectively) exist. Even if we lost our capacity to understand mathematical statements in general and therefore could not find their proofs, according to Frege mathematics (more precisely arithmetic and real analysis) would still be reducible to logic.

> sphere of some general science, then the proposition is a synthetic one.[155]

Clearly, then, the obvious presupposition of Frege's doctrine that the theorems of arithmetic and real analysis are analytic is the derivability of the theorems of elementary number theory and analysis from general laws of logic and definitions. This, the lynch pin of his work, is the doctrine now known as "logicism". Logicism is the doctrine that mathematics - more precisely arithmetic and analysis – amounts to pure logic. There is more to this doctrine than the rather trite idea that mathematical proof must proceed from axioms by means of sheer logic, unaided by geometric analogies or by analogies of any other kind. Moreover, there is more to it than the idea that proof in this sense should be conducted in a *formal* system of *formal* logic so as to achieve the rigour necessary. Rather, logicism is the much stronger, and more striking idea that even specifically mathematical axioms on which to base proofs can be dispensed. *Logic* itself can provide what we think of as the axioms of number theory (i.e. the Peano axioms), as well as of real analysis.

In Frege's view, then, not just proof from an axiomatic basis, but the axiomatic basis itself, need look no further than pure logic. Arithmetic and real analysis *in its entirety* can proceed by using only rules of inference, theorems, and definitions of logic, with no appeal to intuition or experience of any kind, and with *no* assumptions whose truth cannot be proven or shown by logic alone. Intuition, visualization, pictures and diagrams can by no means represent a reason or justification for a mathematical statement to be true. Arithmetic is pure logic. Other thinkers, such as Peano, might have succeeded in reducing e.g. elementary number-theory to the consequences of a set of axioms so

[155] Frege (1884), § 3.

simple in their formulation and import that no mathematician could give a moment's consideration to questioning them. But Frege insisted in pushing back the foundations of mathematics still further by asking after the justification of *these* propositions. His logicist answer is that the justification of these axioms is that they are truths of pure logic.

The thesis that arithmetic and analysis are pure logic is actually stronger than the thesis that they are reducible to truths expressible in purely logical terms. Indeed, the relationship between the latter notion and Frege's thesis that arithmetic is analytic, is a subtle and interesting one. Dummett is illuminating in this regard. He suggests that:

> It would be a mistake, though a natural one, to suppose that Frege's only ground for maintaining the truths of arithmetic to be analytic was his detailed reduction of its fundamental laws to logical truths: for he has, besides, some general arguments, based on the universal applicability he ascribes to arithmetic. *Grundlagen* in fact advances two distinguishable theses about arithmetical truths: that they are analytic, and that they are expressible in purely logical terms.[156]

This passage raises several issues. The first of them is the question of what significance Frege attaches to the notion of "universal applicability". Dummett rightly discerns a connection in Frege's mind between being universally applicable, and logic. In some passages Frege comes close to defining analytic propositions as both generally applicable, that is not governing the domain of spatially intuitiable or physical actuality, and as being reducible to logic[157]. Moreover, Frege does emphasise the

[156] Dummett (1991), p. 43.

[157] Frege (1884), § 14 and § 3.

universality condition is fulfilled too. Arithmetic is not confined to a specific domain of knowledge; it is applicable to the domain of what is countable which is the most comprehensive one, it is the domain of everything thinkable:

> Virtually everything that can be an object of thought may in fact be counted: the ideal as well as the real, concepts as well as things, the temporal as well as the spatial, events as well as bodies, methods as well as theorems; even the numbers themselves can in turn be counted.[158]

So universal applicability is a guide to analyticity in Frege's sense. But Dummett is equally right to warn us to take care in our understanding of what it is a guide to. Nevertheless, his suggestion that "*Grundlagen* in fact advances two distinguishable theses about arithmetical truths: that they are analytic, and that they are expressible in purely logical terms" is puzzling in the light of what we have just seen to be Frege's explicit statement that 'analytic' means reducible to general logical laws and definitions. Indeed, Dummett's remark is all the more puzzling because he maintains, further, that according to Frege, neither of these two principles – analyticity and expressibility in purely logical terms – implies the other. He gives what he claims to be counterexamples to both implications. The first concerns axioms of geometry that contain nonlogical expressions in their formulation. He says that this by itself does not prove geometry to be synthetic since a different system of definitions might show it to be expressible in purely logical terms. The purported counterexample he gives to the implication in the opposite direction is Russell's Axiom of Infinity, a statement to the effect that there are infinitely

[158] Frege's lecture 'Über Formale Theorien der Arithmetic' (given in 1885), the quotation is from Dummett (1991), p. 44.

many individuals. This statement is expressible in purely logical terms alone, but Dummett deems it to be synthetic on the grounds that 'for Russell, neither numbers nor classes - what Frege regarded as logical objects - are individuals'.[159]

The first of these examples is problematic. For Frege does maintain that geometry is not reducible to logic. His grounds for thinking that it is not further illustrates the accuracy of Dummett's observation that Frege has a ground for his claim that arithmetic is analytic other than what he takes to be his successful reduction of it to logic – namely, its universal applicability. For Frege maintains that geometry is synthetic on the ground that it is not generally applicable. He writes:

> Only conceptual thought can in a certain fashion shake free of those axioms, when it assumes a space of four dimensions, say, or of positive curvature. ... For conceptual thought we can always assume the opposite of this or that geometrical axiom, without involving ourselves in any self-contradictions when we draw deductive consequences from assumptions conflicting with intuition such as these. This possibility *shows* that the axioms of geometry are independent of one another and of the fundamental laws of logic, and are *therefore* synthetic.[160] (my emphasis)

By contrast, Dummett's point about Russell's Axiom of Infinity is well taken. There are statements expressible in merely logical terms which are neither logical truths, nor logical falsehoods. Even if some such statement is true, that will hardly suffice for it to be analytic in Frege's sense. It may be true unbeknownst to us, or even unknowably to us. But in

[159] Dummett (1991), p. 43.
[160] Frege (1884), § 14.

envisaging the analyticity of e.g. arithmetic Frege clearly envisages the eventuality that arithmetic can be traced back to *known* logical truths and definitions. So Frege's logicism would not be vindicated were it merely to transpire that arithmetic and analysis can be identified with propositions which are true and whose expression requires none but logical terms. Its vindication requires a *demonstration* that arithmetic and analysis can be identified with propositions which are expressible in purely logical terms *and which can be seen to be logical truths*. The italicised clause in this latter formulation invokes a more or less explicit epistemological element to Frege's logicism which is lacking in the previous one.

Frege's logicism is a provocative and extraordinary thesis. It has both a philosophical, and a mathematical aspect. Both aspects must be considered if it is to be fully appreciated.

2. Methodological background (2) – Ontology and philosophy of language

The nature, import, and motivation of the thesis that mathematics is pure logic depend on one's conception of logic, and of logic's relation to ourselves and to everyday language. Frege's view of these matters is largely implicit in three methodological principles he insists any foundational investigation of mathematics must hold uppermost. These principles are:

Anti-Psychologism: always to separate sharply the psychological from the logical, the subjective from the objective.

The Context Principle: never to ask for the meaning of a word in isolation, but only in the context of a proposition.

The Object/Concept Distinction: always to bear in mind the distinction between object and concept.

I will explain these principles in turn.

Anti-Psychologism - The goal of the first of Frege's principles is to avoid subjective views of mathematics. Frege holds

that mathematical statements are objective, where 'objective' means "what is subject to laws, what can be conceived and judged,..."[161]. It also means being independent of our constructions, representations or beliefs. If numbers, as Frege points out, were representations, mathematics would be a part of psychology. But mathematics is no more a part of psychology than is e.g. physics. Arithmetic is about numbers as objectively and independently existing objects; it is not about the way we might represent these objects to ourselves, in the same way in which astronomy is about planets. Nor is it about the way we might see these objects. This viewpoint, of course, is that of the realism about truth value and ontology argued for in Part 1 of this book.

It is also important that Frege has none of the nominalist qualms which philosophers of mathematics have frequently manifested. For him, being objective does not mean necessarily being spatio-temporal located and concrete. He gives the examples of the equator, the earth's axis, and the centre of mass of the solar system.[162] These objects, he maintains, are not physical.

And yet, the *objectivist* element to Frege's thought should not be overplayed. In Section 26 of *Grundlagen*, a section that has caused much controversy, he asserts:

> for what are things independent of the reason? To answer that would be as much as to judge without judging, or to wash the fur without wetting it.

[161] Frege (1884), § 26.

[162] Frege (1884), § 26. One might take issue with Frege's examples. Both the equator and the centre of mass of the solar system have spatio-temporal location, and it is by no means clear that these objects are not concrete (and thus not physical objects). One might recall from the discussion in Chapter 2 above, that one of the foremost contemporary nominalists, Hartry Field, offers an ontology of space-time points as *an alternative* to a platonist ontology of abstract objects.

How is this assertion to be understood? Out of context, one might perhaps interpret it as recommending the mind-dependence of mathematical truth. However, an interpretation of this kind does not gel with the main body of Frege's work. The only interpretation consistent with other Frege's views is a reading on which the assertion is directed against Kant's advocacy of a *transcendental* realm of utterly unknowable "things-in-themselves" (i.e. "noumena"). Read in the broader context of Frege's thought, the passage quoted is an assertion to the effect that reason – and hence logic, and hence rational or logical thought – is the arbiter of all things. Everything that has an existence independent of the mind (in the sense that it could have existed even if no minds had existed), still has to obey the general laws of logic, and hence of rationality, and of thought.[163] On this view, if knowledge of a certain object that this object has a certain feature is grounded in logic, or rational thought, this knowledge *is* knowledge of the thing-in-itself. *A fortiori*, Frege likewise has no sympathy for the excesses of the pre-Kantian rationalist tradition's pandering to the hypothesis of God's omnipotence, i.e. whereby it was maintained that God could have constructed a world in which the laws of logic are other than the laws of logic are actually, and, hence, that a real question arises as to whether the laws of logic are anything like what we take them to be.

The Context Principle – The import of Frege's second principle is that we cannot analyse words and prescribe them meanings in isolation, outside of and independently of their role in propositions. Frege's target here is the atomism which had been a presupposition of the entire empiricist tradition which went before him. That tradition had ignored the need

[163] According to some interpretations Frege's assertion that things have no reality independent of reason is a sign of Frege being as a matter of fact an anti-realist. However, while this assertion taken in isolation might suggest such an interpretation, the interpretation it is certainly not possible in the context of the overall Frege's theory.

for a systematic account of the contribution a word makes to the meaningful propositions in which it occurs. Its attitude was, so to speak, to look for the meaning of a word in isolation, and then, having hit upon a likely candidate, to *assume* that this meaning could be combined, unproblematically, with other meanings so as to produce a proposition. Frege's context-principle is diametrically opposed to this tradition. Frege recognises – and this is one of his lasting legacies to philosophical logic and philosophy of language – that the primary bearer of meaning is not a proper name, and still less a word. Rather, it is the atomic sentence: the shortest unit of language with which it is possible to perform a significant speech-act. Obviously, independently of a suitable completing *linguistic* context, merely uttering a name, "Tony Blair" is utterly meaningless. By contrast, uttering a sentence, "Tony Blair is at the beach", even without any context, and even if the utterance is socially and psychologically bizarre is perfectly meaningful. Indeed, we think of an utterance this kind "out of the blue" as socially or psychologically bizarre precisely because we know what that it is the assertion that Tony Blair is at the beach.

A further rationale Frege gives the context principle is that failing to conform to it is bound to involve a failure to conform to the principle of anti-psychologism too. For, as the empiricist tradition had hitherto illustrated perfectly, looking for the meaning of words in isolation is bound to result in our confusing that meaning with one of our internal representations, or "ideas", and, hence a flouting of the distinction between psychology and logic, between the subjective and the objective. Again, Frege rightly insists that meanings are themselves objective: since the meaning of one person's speech can be grasped by any other speaker of the language, linguistic meaning is public. But the "ideas" or internal representations so beloved of the empiricists, which individuals associate with words – think now of the ideas conjured up by the word "ostrich" as you read – are *subjective*. Precisely because different people attach different ideas in this sense to a word such as "ostrich", these ideas can have nothing to do with the word's meaning. Rather, its meaning is the contribution the word makes to the publicly graspable meanings, or, in Frege's terminology, "thoughts", that are expressed by grammatical sentences in which the word "ostrich" occurs. The impact of this perspective on Frege's reflections on the meaning of words and phrases expressing fundamental mathematical concepts, such as "the number 1", or "addition", or "natural number", is predictable, and can hardly be exaggerated.

The Object/Concept Distinction - In the third principle Frege asserts that there is a sharp and fundamental distinction between two kinds of entity: object and concept. There is no hint of psychology in this distinction: it is logical, and ontological. And in Frege's view it is also exclusive: no concepts are objects, and no objects are concepts. Intuitively, an object is a thing, either material or not, concrete or abstract, while concepts are "items very much like what the logicians of the Middle Ages would have called "universals" or Plato would have referred to as "forms", things that all objects of a certain kind have in common. (Some near-synonyms are: property, quality, characteristic, attribute.)"[164] Formally, Frege defines objects as referents of singular terms and concepts as referents of predicates. In a sentence, the subject refers to an object, while the predicate refers to a concept. A proper name, that is the name of an object, cannot be used as a predicate in any sentence.

While one might readily acknowledge that Frege's distinction between concept and object is no doubt of some significance in general metaphysics, or philosophy of language, one

[164] Boolos, 'Gottolob Frege and the foundations of Arithmetic', in Bolos (1998), p. 149.

might wonder why it should be of any great importance to the philosophy of mathematics. But it is of the utmost importance, both for the negative arguments that led Frege to reject earlier doctrines, and for his own alternatives to them. Frege's employment of the distinction is critical at the very outset. Many of Frege's predecessor's analysed commonplace assertions of cardinality, such as "there are five apples on the table", as attributions of a property (concept) –numbering five, or being five membered or whatever – to an object – the aggregate or set of apples. But Frege argues that this is a fundamental misunderstanding. Assertions of cardinality of this kind in fact attribute a concept *to a concept*.

While I consider what this Fregean doctrine amounts to further below,[165] I mention it now in an attempt to indicate why Frege attaches such importance to the distinction between concepts and objects. His insistence on it in the *Grundagen* attracts criticism from Kerry, who argues that 'being an object' and 'being a concept' are not exclusive properties. Frege rebuts Kerry's criticism, and insists on the exclusivity of the distinction.[166] To be fair to Kerry, it has to be admitted that merely characterising objects as the referents of possible proper names, and concepts as the referents of possible predicates, leaves open the possibility of an entity which is both the referent of a proper name *and* the referent of a predicate. And further argument is needed to discount this possibility. Of course, one might reasonably think with Frege that it never happens that a predicate is substitutable for a proper name in a sentence so as to say exactly the same thing. But while one might take this to indicate that there are no proper names the referents of which are also the referents of predicates (and vice-versa), couldn't this failure of substitutivity be a mere

[165] See Chapter 8.

[166] See Frege (1892).

quirk of Indo-European grammar? Moreover, there are cases that might seem to be counter-examples to the exclusivity of Frege's distinction. While we say that something is green or a mammal, we might also say that something is Alexander the Greek, or the planet Venus. Doesn't this much illustrate at least that a proper name can be substituted for a predicate, and hence an object for a concept, without impugning the meaningfulness of the whole proposition, i.e. the implication being that at least some objects are concepts after all? However, Frege accommodates cases of this kind to the exclusiveness of his distinction by arguing against the supposition that e.g. "Hesperus is Venus" can be obtained from "Hesperus is beautiful" by substituting the name "Venus" for the predicate "beautiful". He deems this supposition a confusion, and an equivocation. These two sentences do not contain a common part, "Hesperus is ..."; namely, in the former case the term 'is' is a mere copula, a sign which serves merely to conjoin the subject term "Hesperus" with the predicate "beautiful", while in the latter case the term 'is' is used as the equality or identity sign which is a commonplace in arithmetic. The identity sign, 'is' is *itself* a (two-place) predicate referring to a concept. As Frege puts it (using a slightly different example):

> In the sentence 'The morning star is Venus', 'is' is obviously not the mere copula; its content is an essential part of the predicate, so that the word 'Venus' does not constitute the whole of the predicate. One might say instead: 'The morning star is no other than Venus'; ... and in 'is no other than' the word 'is' now really is the mere copula.[167]

[167] Frege (1892a), p. 44.

3. Rigorous implementation of the methodological principles: formal languages

The three principles just adduced led Frege to develop a formal language in which his view of the nature and objectivity of meanings is manifest. In *Begriffsschrift*[168], we find "translated" the concept-object distinction into the language of second-order predicate calculus. This is a formal language comprising variables $x_1, x_2, \ldots$ that refer to objects, and variables $f_1, f_2, \ldots$ that refer to functions. In mathematics a function f (or a mapping) from a nonempty set X to a nonempty set Y (not necessarily different from X) is a rule or relation which puts every member (or element) x of the set X into correspondence with one and only one member y from the set Y.[169] The notation is $y = f(x)$: it expresses the fact that the function f maps x into y. This can be put otherwise by saying that y depends on x; accordingly, x is called the independent variable and y the dependent variable. Frege follows this practice.[170] He calls the object x of X the 'argument' of the function f, while $f(x)$, the object y of Y to which the function f maps x, is the 'value' of the function. What he calls the "course of values" of a function f is in modern terms the graph of a function, that is the set of all ordered pairs $(x, f(x))$; Frege uses the notation $\dot{\varepsilon}f(\varepsilon)$.

[168] Frege (1879).

[169] Functions are therefore specific sorts of relations. In relations any element x from the first set X can be put into correspondence with one or more elements y from the second set Y. 'Being higher then' among people is an example of a relation that is not a function because one person x can be higher and therefore in relation (or put into correspondence) with more then one person.
It has to be specified though that the above definition of function is the definition of a single-valued function and, unless it is mentioned differently, that is what is meant by the term 'function'.

[170] See Frege (1879), page 16.

Frege takes functions to be entities that are what he calls "unsaturated", in that, in containing the independent variable x, they are waiting to be completed or saturated by a value of the variable. Similarly, the unsaturatedness of functions is mirrored at the linguistic level of expressions, such as '3x+5', 'x^2-2', 'being the father of x' and so on. These expressions contain an empty place (that of x). They are waiting to be completed by means of proper names - numerals or, as in the last example, with proper names of people. By contrast, objects comprise everything that is not a function; objects are "saturated". They are complete in themselves, and unlike functions do not await completion by something else. Again, this feature of the ontological level is mirrored in the linguistic one. Objects are referred to by expressions that are saturated, with no empty place to be filled in. These expressions are proper names.[171]

This ontology of (saturated) objects and (unsaturated) functions might seem to leave concepts out of the picture, but it does not. For concepts are defined as a sub-species of the more general category of functions. The category of expressions referring to functions includes expressions like '$x^2=1$'.[172] An expression such as this has a peculiar characteristic. With respect to any value of x by which we might complete it, the result is an expression – a complete sentence - that is either true or false. For example, for x=2 the expression is false ($2^2 \neq 1$), while for x=1 it is true ($1^2=1$). Hence the function the original expression referred to is special, in that it takes numbers as arguments and gives as values either The True or The False. It is functions of this kind that Frege terms "concepts". Concepts are a sub-species of the category of functions. They are functions that map their arguments exclusively to truth

[171] See Frege (1921), pp. 21-41.

[172] See Frege (1921).

values: either The True or The False. A function F(x) can therefore be equal just either to The True or to The False, for every argument x. An example of a function that is a concept might be: the function 'being strictly greater than zero' or 'having lungs'. The first one has as its arguments numbers, where the function, that is concept, maps any positive number to The True, and zero as well as negative ones to The False. The second example has as arguments living beings, and maps all the mammals to The True and e.g. lobsters to The False.

Since the values of such functions are expressed by saturated expressions they are objects. On the other hand the values are just The True or The False. It follows that The True and The False are objects too.[173] Declarative sentences, which we think of intuitively as being true, or false, are therefore proper names for truth values. So, for example, '$2^2 = 1$' is a proper name for The False, while '$1^2 = 1$' is proper name for The True.

The arguments that a concept F maps to The True are those objects that, as Frege would say, "fall under the concept F". Human beings fall under the concept 'having lungs', while e.g. fish do not. Natural numbers all fall under the concept 'strictly greater than zero', while, e.g. -5 does not.

That the informal grammatical concept-object distinction is equivalent to the formal logical-mathematical distinction is evident. Terms referring to functions can be identified with all and only those expressions that play the role of predicates in sentences. No subject in a sentence can be treated as the name of a function, since it always refers to an object.[174] Subjects in sentences refer to objects, which are saturated, and

[173] See Frege (1921).

[174] The immediate result of this classification, noted by Frege himself, is the truth – somewhat paradoxically, of sentences like e.g. "The concept 'horse' is not a concept".

therefore cannot denote functions, which are unsaturated. Predicates refer to functions and hence cannot stand for objects. Predicates are not proper names. Functions are not objects.

8

Frege's Logicism (2) – Platonism and the Logicisation of Arithmetic

Having outlined the framework constituted by Frege's ontology and philosophy of language, we are now in a position to see how he applies this framework to the specific problem of mathematics. Frege focuses on arithmetic and analysis. Indeed, we have seen that his view of e.g. geometry is very different. And since the main conceptual issues raised by his logicism are apparent in the former case, I will focus still further on arithmetic itself. Before going into the technical detail of his logicist treatment of arithmetic, however, it is as well to begin with a puzzle. The reader may well wonder how Frege's logicism can possibly count as a *platonist* philosophy of mathematics. Doesn't the supposition that e.g. arithmetic amounts to pure logic preclude the platonist doctrine that arithmetic is the study of certain abstract objects? How can *pure logic* be a study of such objects? Without answers to these questions, Frege's philosophy of mathematics cannot but appear a muddle.

1. Ontology and logic

It is a popular view in contemporary logic, which goes back to Kant, that there is no ontology in logic, that there are no logical objects. Frege, on the contrary, beliefs that logic does have an ontology. He asserts both that there are mathematical objects and that such objects can be defined in terms of logic. According to him, realism in ontology and logicism are not in conflict with each other. In particular, platonism is consistent with logicism. For logic objects are abstract, non spatio-temporal located, they are objects whose existence is objective and independent. He argues that this ontology makes logic neither less topic-neutral nor general, that is universally applicable.

The key to understanding Frege's view that logicism and platonism are reconcilable is intimacy of the connection, noted in the previous chapter, Frege establishes between fundamental ontological categories – in his view object and concept – and the syntactic categories – subject and predicate – in terms of which formal languages, and hence logic, are most clearly and rigorously developed. More importantly, it is not just that these syntactic categories are held to mirror the ontological ones. Rather, the ontological categories are themselves linguistic, in a sense. For the syntactic categories are prior to ontological ones:

> it is by reference to the syntactic structure of true statements that ontological questions are to be understood and settled.[175]

In Frege's view, fundamental ontological questions – what exists, and what sorts of things exist – are ultimately logico-linguistic. For him, there is no possibility of a situation in which there are no genuine objects even though there is a syntax of our language with referential terms making apparent reference to these objects and appropriate statements containing these terms which are true. Questions about the existence of an object are actually questions about whether certain proper name has a referent and propositions about the object are *objectively* true:

> The question whether '*a*' refers, if, like 'Vulcan' or 'the largest odd perfect number' '*a*' is part of the language, is intelligible only in terms of our

[175] Wright (1983). p. 25

> understanding of statements, *par excellence* statements of identity containing it.
> The tendency which Frege is opposing would allow that even if, in terms of that conception, an appropriate such statement is true, and even if '*a*' functions, by all syntactic criteria, just like a singular term, it may be that '*a*' has no actual reference. For Frege this is a confusion. We have no grip on any *further* question about '*a*''s claim to reference.[176]

In § 29 of *Grundgesetze,* Frege gives the condition for an expression to have a reference:

> (i) an expression for a first-level function of one argument has a reference provided that the result of inserting a referential term in its argument-place is always again a referential term;
>
> (ii) a singular term ('proper name') has a reference if
>
> (a) the result of inserting it in the argument-place of a referential expression for a first-level function of one argument is always a referential term; and
> (b) the result of inserting the given term in either of the argument-places of a referential expression for a first-level function of two arguments is a referential expression for a first-level function of one argument;
>
> (iii) an expression for a first-level function of two arguments has a reference if the result of

[176] Wright (1983), p. 172.

filling both of its argument-places with referential singular terms always has a reference;

(iv) an expression for a second-level function which takes a first-level function of one argument as its sole argument has a reference if the result of inserting in its argument-place a referential expression for a first-level function of one argument always has a reference.

The upshot of this is that once we know that a term *t* is a singular term, or in Frege's terminology a 'proper name', and that statements of identity containing this term are (objectively) true, we can infer that there is an object, x, such that x is the referent of *t*.

Clearly, from this perspective, realism about ontology vis-à-vis arithmetic requires no more than two results. Firstly, that numerals are, *syntactically*, genuine singular terms. Secondly, that numerals occur in identity statements that are objectively true. But once it is seen that meeting these two conditions suffices for realism – and hence platonism, on the natural further assumption that the referents of numerals cannot be concrete objects (for reasons given in Chapter 4) – the apparent tension between platonism and logicism collapses. For it can be no part of logicism to deny the thesis that, *syntactically*, numerals function as singular terms, or that numerals occur in identity statements that are objectively true.

2. The implementation of logicism: The case of arithmetic

If one's aim is to prove that the theorems of arithmetic are analytic, and hence reducible to pure logic, the first step must be to define the basic concepts of arithmetic exclusively in terms of the concepts of logic. The most elementary arithmetical concept is that of 'natural number', that is, the concept of

a positive integer. This is (at least) the concept of something which provides a possible answer to the question "How many?". As Frege himself says, "on the outcome of this task will depend the decision as to the nature of the laws of arithmetic"[177]

How one sets about defining the concept of natural number is bound to be influenced by the ontological category to which natural numbers are taken to belong. As we have just seen, Frege's doctrine of the priority of syntactic categories to ontological ones leads him to suppose that numbers are objects if numerals are singular terms and occur in true identity statements. But *are* numerals singular terms? Reflection on statements of cardinality – i.e. statements of the form "there are n F's", or "the number of F's is n", or "the F's were n (in number)" had led some of his predecessors to suppose that they are not. Consider a (somewhat officious) statement like "Those found drunk in charge of their vehicles were five". Here, "five" might seem to occur as a predicate, the form of the statement being the same as that of e.g. "Those found in charge of their vehicles were abusive to the policeman who charged them". This analogy had led various philosophers to suppose that just as the latter statement attributes a property to certain objects, so too does the former statement that attributes a property to a special kind of object – an aggregate comprising the five drunkards.

However, Frege argues forcefully against this view. In his view a statement of cardinality does not attribute a property of a (peculiar) object. Rather, it attributes a property to a concept – or, better, it asserts of some concept that it falls under some other. In his view talking about one thousand leaves is different form talking about one thousand leaves being

[177] Frege (1884), § 4.

green.[178] Being green is a property attributable to each leaf, while it is certainly not the case that being one thousand is similarly attributable. Furthermore, one and the same phenomenon can be counted in different ways: something can be counted as one symphonic orchestra, 3 sections of instruments (wind, string and percussion), 23 different instruments or 73 musicians.

Particularly problematic for the view that a statement of cardinality of the form "there are n F's" ascribes a property to an object is the case in which n is zero. When we say that the planet Venus has zero satellites, we are certainly not ascribing a property to some satellite(s). Frege takes this example to be definitive. We have no choice, he suggests, but to construe this assertion as an assertion about the *concept* 'satellite of Venus'. Since the point is principled, assertions of cardinality in which n is greater than zero must be treated similarly. In the examples above, 'one thousand' is employed to attribute a property to the concept 'leaves of the tree', 'three' to attribute a property to the concept 'sections in a symphonic orchestra', and so on. Attributions of cardinality invariably ascribe properties to concepts (or, better, in Frege's terminology, they assert that one concept falls under another).

At this point, one might wonder whether Frege has proved too much. We set out to discover whether numbers are objects, and, hence, from Frege's perspective, whether numerals are singular terms. We have now seen Frege who has argued that the statements of cardinality in which numerals paradigmatically occur are assertions about a concept to the effect that this concept falls under some other. Should we not conclude at this point that in Frege's view numerals are *not* singular terms? No, we should definitely not; we must be cautious here. To maintain that in the statement "There are 73

[178] Frege (1884), § 22.

musicians in the orchestra' the number '73' is being employed to assert of a concept that it falls under another is not say that the number '73' itself refers to a concept. On the contrary, Frege is adamant that numbers are not properties of concepts. He writes:

> I have avoided calling a number like 0, 1 or 2 a *property* of a concept. The individual number, as being self-subsistent object, appears precisely as a mere part of the predicate.[179]

Numbers cannot be treated as properties of concepts, or in Frege's terminology, as concepts that other concepts might or might not fall under. Concepts are referred to be predicates. Hence, for numbers to be concepts, numerals would have to be predicates. But of course they are not. Numerals do not behave like ordinary predicative expressions at all. Rather, syntactically, they behave like paradigmatic singular terms. We say 'Spiderman is brave' and 'Batman is brave', therefore 'Spiderman and Batman are brave'. But we can hardly say 'Spiderman is one' and 'Batman is one', therefore 'Spiderman and Batman are one'. Besides that, with number words and expressions only the *definite* article can be used: "*the* number one", "*the* number two", and so on while their plurals do not exist. The moral is that numbers are *parts* of the concepts other concepts are said to fall under in attributions of cardinality. The numeral "73" is employed as *part* of the predicate in the statement "there are 73 musicians in the orchestra". But the numeral itself is a proper name, and the number it refers to an object.

Having established the basic ontological category to which numbers belong, Frege's next task, and the most important one, is to say *which* objects the natural numbers are. Since a

[179] Frege (1884), § 57.

number is an object, it is required to establish preliminarily how to determine if, given a number *a*, another object *b* is identical to *a* or not. For objects are subject to a determinate identity principle: each object is identical to itself and to no other. That means, in the light of Frege's context principle (see previous chapter), that it is necessary to give a meaning to the proposition: "The number belonging to the concept F is identical to the number belonging to the concept G". In other words it is necessary to establish when two concepts F and G are equinumerous. In defining equinumerosity Frege expresses in terms of logic the standard mathematical definition of equinumerosity of sets. Two sets, A and B, have the same number of elements[180] if it is possible to establish a bijection[181] between the two sets. In layman's terms, given two bags, A and B, of pebbles and asked to say if the two bags are equinumerous we would solve the problem by taking one pebble from the first bag and one from the second, then a second pebble from the first bag and the second one from the second bag[182] and so on. In other words, we would put into correspondence the pebbles from the first bag with those from the second one. Eventually, there would be three possible outcomes:

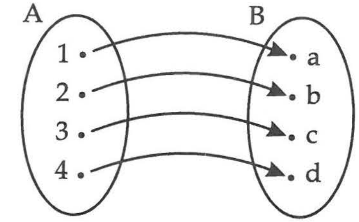
1. The two bags are equinumerous, which means that every pebble from the first bag corresponds to one pebble from the second bag and the other way round.

[180] We say that sets that have the same number of elements have the same cardinality.

[181] A bijection is a function from the set A to the set B such that two different arguments are mapped to different values and every element form the second set is a value of an x from the first set. It is also called one-to-one correspondence.

[182] Given the supposition that the bags are not empty. If there are no pebbles in the two bags, the bags are equinumerous anyhow.

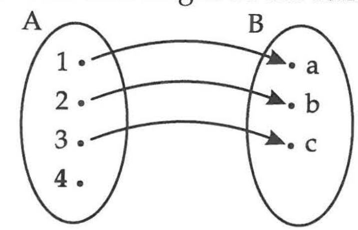
An example of such a case might be presented in the following way[183]:



In such case we say that a bijection has been establish between the two bags.

2. The first bag has more pebbles than the second one: at some point when we take one pebble from the first bag while we have run out of those from the second one. Since the goal is to determine just if the given bags have the same number of pebbles, it is achieved as soon as we have no pebbles left in one bag while there is at least one pebble left in the other one.

An example of this case might be the following diagram:



In this case there is no bijection between the two bags; the two bags are not equinumerous.

The third possibility, the one in which the second bag has more element, is analogous to the second one. Neither in this

[183] The diagram shows the case in which there are at least four pebbles in every bag; obviously it is not necessarily so. The pebbles in one bag are marked with numbers, the one from he other bag with letters just in order to distinguish them.

case there is a bijection between the two sets nor are they equinumerous.

While the concept of equinumerosity is clear enough, if mathematical theorems are to seen to be analytic, every definition or theorem has to be expressed in terms of logic. Clearly, then, Frege is not in a position to appeal to a primitive concept of equinumerosity. And he does not do so. He articulates the notion of one-to-one correspondence in purely logical terms. In *Grundlagen*[184] he defines equinumerosity as follows:

F and G are equinumerous if there is a relation ϕ such that:

1) every object falling under the concept F is ϕ-related to a unique object falling under the concept G and

2) every object falling under the concept G is such that there is a unique object falling under the concept F such that is ϕ-related to it.

Two conditions are therefore required: the existence of a relation and such a relation being one-to-one.

It means that for every such relation ϕ the following two conditions must be fulfilled[185]:

1) $d\phi a \wedge d\phi b \Rightarrow a = b, \quad \forall\, d, a, b$

2) $d\phi a \wedge b\phi a \Rightarrow d = b, \quad \forall\, a, b, d$

Having defined equinumerosity, Frege is now in a position to assert the following criterion of identity formally:

3) The number of Fs is identical to the number of Gs

if and only if

the concept F is equinumerous with the concept G[186].

[184] See §§ 71, 72.

[185] These two conditions are identical with the two mentioned when defining a bijection or one-to-one correspondence. See footnote 181 above.

[186] Frege (1884), § 73.

In symbols:

$\forall F\ \forall G\ (\ n(F) = n(G) \Leftrightarrow F \approx G\ )$

taking n(...) as abbreviating 'the number of ...s', and $F \approx G$ to abbreviate 'F is equinumerous with G'[187].

This principle has been called 'Hume's principle' by Boolos (in his 'The consistency of Frege's *Foundations of Arithmetic*').[188] Boolos gave it this title because it recalls a remark in Hume's *Treatise* (Book I, Part iii, Section 1, par. 5), and because Frege quotes Hume in *Grundlagen*[189]:

> When two numbers are so combin'd, as that the one always as unite answering to every unite of the other, we pronounce them equal...[190]

Is Hume's principle the criterion of identity for numbers that Frege seeks? No, it is not. Frege is adamant that Hume's Principle does *not* suffice to clarify the identity conditions of the natural numbers. His reason for saying so is that while it serves perfectly well to give meaning to *some* numerical identities, it does not suffice to give meaning to them all. Although it determines the truth value of identities of the form: 'the number of F = the number of G', for any two concepts F and G, it does not determine the truth value of sentences of the form: 'the number of F = x', for arbitrary x. The consequence Frege draws from this limitation is that:

[187] '$F \approx G$' or 'F eq G' or '$F\ 1\text{-}1_R\ G$'.

[188] In Boolos (1998), pp. 183-202.

[189] § 63.

[190] In FOM (Foundation of Mathematics) discussion group on the Internet, there was a debate during March/April 2001 if Hume's principle was the appropriate naming of the cardinality principle. Some members of the list (for example Tait) thought it was not since the naming was due to a misreading of Hume that went back to Frege. It seems to me, on the contrary, that Hume's passage that amounts to the idea that it is possible to determine when two positive integers are equal by writing them out as n=1+1+1+...+1 and then comparising units is, at its roots, the idea of one-to-one correspondence even though confined to finite domains.

> ... we can never - to take a crude example - decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that conqueror of Gaul is a number or is not.[191]

The problem Frege raises here, whereby Hume's principle seems to leave the truth value – and hence the meaning - of so-called "mixed" identities undetermined is now called the 'Caesar problem'. We will see in the next chapter that contemporary "neo-logicists" draw a very different moral from this problem than does Frege himself. But for the moment let us concentrate solely on Frege's own response to it. The moral Frege draws from the difficulty is that instead of being appealed to as basic, the identity condition embodied in Hume's principle should be implicit in an identity condition which is both more general and more fundamental. The only way to hit on such a condition, Frege suggests, is to provide *explicit* definitions of the natural numbers. That is, it is necessary to define 'the number belonging to the concept F' or, simply, 'the number of Fs'.

To define explicitly the number of Fs, Frege appeals to the notion of the *extension* of a concept. This comprises all objects that the concept applies to. In terms of Frege's logic, the extension of a concept is the course-of-values that records which objects the concept maps to The True. In particular, he suggests that

The number belonging to the concept F (that is the number of Fs) is the extension of the concept "equinumerous with F".

Or, alternatively, that

The number of Fs is the extension of the concept "being a concept equinumerous with F".

[191] Frege (1884), § 55.

The upshot of this explicit definition is that the number of Fs is a collection/class of concepts. The concept "being a concept equinumerous with F" is a second-order concept comprising in its extension not objects, but concepts; these being the concepts G such that the objects falling under G are in one-to-one correspondence with the objects falling under F. Concepts such second-order concept has in its extension are first-order concepts that is, concepts containing objects in their extensions.[192]

According to the definition, for example, the number of the concept 'satellite of the planet Earth' is the extension consisting in (first-order) concepts that hold of exactly one object: concepts like 'being the positive root of the equation x^2-1 = 0', 'being the nearest planet to the Sun', 'being the capital city of Egypt' are all members of that extension and the cardinal number 1 is identified with it.

Numbers are peculiar sort of extensions though, in the sense that one cannot possibly be more comprehensive that others[193]. Numbers are objects; one can be smaller that some others but cannot be included in any of them. 3 < 4 but the number three is not in any way part or subcollection/subclass of the number four.

Frege's definition of numbers is usually glossed as a definition of numbers as sets. Namely, it is analogous to the mathematical definition of numbers as equivalence classes: the number *n* is the set of all sets containing *n* elements. The domain of sets is divided in groups - or technically equivalence classes - so that all sets that are equinumerous belong to the same group, that is the same equivalence class.[194]

[192] Frege (1884), § 53. See also Frege (1892a).

[193] Frege (1884), § 69.

[194] Such groups are called equivalence classes because they are obtained by introducing in the domain of sets the relation ρ: AρB ⇔ sets A and B are equinumerous.

What is essential for Frege is that his definition of cardinal number is analogous to the mathematical one while expressed with and reduced to (two) purely logical terms: relation and extension. He takes this to constitute a proof that the basic mathematical terms are reducible to logical ones.

3. The inconsistency in Frege's system

Frege presents his results strictly translated in logical symbols as well as the extension to the real numbers in the *Grundgesetze der Arithmetik* (or *Basic Laws of Arithmetic*).[195]

A problem nevertheless appears in Frege's work: the inconsistency of his system presented in the *Basic Laws of Arithmetic*. The inconsistency is due to the fact that Frege's theory of extensions is tantamount to naïve set theory, the inconsistency of naïve set theory being well known.

In order to axiomatize and stipulate a theory of extensions Frege introduces the now notorious Basic Law V[196]:

$$\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha) \Leftrightarrow \forall x(f(x) = g(x))$$

It asserts that the course-of-values of the function f is identical with the course-of-values of the function g if and only if, with respect to every object x, f and g map x to the same object.

If the functions f and g are the concepts F and G, the Basic Law V has the special case:

$$\grave{\varepsilon}F\varepsilon = \grave{\alpha}G\alpha \Leftrightarrow \forall x(Fx \leftrightarrow Gx)$$

The relation ρ is an equivalence relation which means that it is:

i) reflexive: $A\rho A,\ \forall A$

ii) symmetric: $A\rho B \rightarrow B\rho A,\ \forall A, B$

iii) transitive: $A\rho B \wedge B\rho C \rightarrow A\rho C,\ \forall A, B, C$

and therefore the classes obtained with such a relation are equivalence classes.

195 Two volumes, 1893 and 1903.

196 Frege (1893), § 20.

It asserts that the extension of the concept F is identical with the extension of the concept G if and only if every object that falls under F falls under G and conversely, every object falling under G falls under F.

Even though at first sight it seems unproblematic, Basic Law V is inconsistent. Frege himself, after receiving in 1902 a letter from Russell pointing to the paradox due to the Basic Law V, adds an Appendix to the second volume of his *Basic Laws of Arithmetic* in which he describes two ways of deriving a contradiction from the Basic Law V. Both of them follows from a corollary to the Basic Law V, viz. that every concept has an extension.

In one respect, of course, the significance of Russell's letter cannot be underestimated. What feature of a foundation for mathematics could be worse than logical contradiction?! However, the fact remains that many of Frege's achievements survive the proof of inconsistency intact. His contribution to philosophical logic and philosophy of language pervades contemporary thought in the area, and it remains significant for the philosophy of mathematics: no one can reflect on the nature of mathematical judgement, or proof, efficiently in ignorance of Frege's treatment of the subject. And of course we owe to his efforts modern logic. Moreover, some of his purely formal results are lasting. In particular, he proves the surprising result that the Peano axioms can be derived in second-order logic with the addition of Hume's principle alone. For some this fact will be of only marginal interest. Faced with the contradiction in Frege's system, they insist on demarcating set theory from logic in a way Frege tries to resist. A refinement of naïve set theory is readily formulated in which the contradictions of naïve set theory are not reproducible – ZFC will do – and as much mathematics as we could wish for is articulable within such a theory. To be sure, the axioms of the theory are not logic, but they are compelling enough, and it is reasonable to think them consistent. Frege's logicism was a fruitful research program, right enough, but it is now a spent force. Contemporary platonists should seek to solve the problems of indeterminacy and of epistemology which beset their theory without appealing to the relative lucidity and comfort of pure logic.

Others, however, see this attitude as unduly defeatist. For them, Frege's derivation of the Peano axioms from Hume's principle in second logic is more than a technical nicety. It is a result that promises to yield the result he yearned for without the detour through an explicit definition of natural numbers that proves inconsistent. These are the "neo-logicists". As we have seen, Frege had his reasons for taking this detour. For the neo-logicists, however, these reasons were misguided. It is the detour, not logicism itself, which was Frege's fatal mistake.

9

Neo-Logicism (1) – The Programme

In this chapter and the next one I explain and evaluate "neo-logicism's" attempt to salvage the insights of Frege's logicism from the havoc wrought by the contradictoriness of Basic Law V. In effect, neo-logicism attempts two interwoven tasks. Firstly, it attempts to vindicate the spirit, if not the letter, of the basic doctrines of Frege's logicism, by developing a systematic treatment of arithmetic and analysis that *approaches* the requirements of Frege's doctrine while avoiding inconsistency. Secondly, it attempts to consolidate this doctrine's approach to the ontological and epistemological difficulties that have always dogged platonism.

The present chapter is a critical exegesis of two issues which are central to the neo-logicist's programme. Having briefly explained the essential features of neo-logicism, I turn first to a matter of detail: the precise formulation and rationale of Hume's Principle. Then I turn to what is in many ways the crux of the matter: the ontological and epistemological significance of so-called "abstraction principles" (of which Hume's principle is but one).

1. The essential features of neo-logicism

As we saw in the previous chapter, in the *Grundlagen* Frege proved an exciting result: the derivability in second order logic of Peano's axioms from Hume's principle.[197] However, he himself never thought of this result as important. This was not because of the logicist goal he had set himself of showing

[197] They were not the Peano (or Dedekind-Peano) axioms but others equivalent to them. Frege sketched the derivation in Frege (1884), §§ 73-83 and hereinafter in Frege (1903).

arithmetic to be analytic – and hence reducible to logic and definitions. To be sure, Hume's principle could be thought of as no more than an "implicit" definition at best: unlike "explicit" definitions, it did not permit the elimination of singular terms in all contexts. But it was only in later years he became dismissive of definitions other than explicit ones, and at the time of the *Grundlagen* at least he was prepared to countenance implicit definition. Rather, the weakness in his eyes of Hume's principle was its vulnerability to the Caesar Problem: in his view, it gave no sense to "mixed" identity statements involving numerical singular terms, and to that extent failed to meet prior constraints on acceptable definitions. It was this consideration which led him to seek an explicit definition of "the number of F's" and, of course, of numerical singular terms. In turn, this led him into the theory of extensions (his definition of e.g. the former being 'the extension of the second-order concept: "being equinumerous with F"'), and, hence, to the disastrous Basic Law V.

Once the inconsistency of Basic Law V became known to Frege, he quickly abandoned his logicism programme as a whole.[198] Neo-logicisists, however, advocate a more measured response. They observe that the explicit definition of numbers, and hence the theory of extensions, and therefore the inconsistent Basic Law V, served no purpose, in effect, other than the derivation of Hume's Principle. Since Frege himself showed that Hume's Principle alone suffices for the derivation of the Peano axioms in second order logic (this result is known as "Frege's theorem"), the *formal* requirements of arithmetic at least can therefore be met, without inconsisten-

[198] Frege did, in his latest years, tried to base arithmetic on geometry but I have been concentrated in the thesis exclusively on his logicist programme as a platonistic one.

cy, by simply adding Hume's principle to the second-order logic as a supplementary axiom.[199]

According to Frege's neo-logicist followers, then, his logicism was correct in all fundamental respects except for two related points at which his judgement went awry. Firstly, he overestimated the significance of the Caesar problem. And secondly, he underestimated the significance of his derivation of the Peano axioms from Hume's principle in second order logic, and of the possibility of *grounding* the claim that arithmetic is analytic in Hume's principle.

In effect, neo-Fregeanism consists in two main claims:

(1) The logical claim: Hume's principle plus second-order logic together comprise a consistent system from which the Peano axioms, and hence arithmetic itself, can be derived. Moreover, Hume's principle is properly viewed as *a conceptual truth* having a similar epistemic status as does logic itself. As Hale puts the thesis:

> Whether this fact supports any kind of logicism about arithmetic depends, of course, on the status of Hume's principle. Boolos, along with many others, denies - plausibly, in my view - that it can be regarded as a *truth of logic*. Further, Hume's principle cannot be taken as a *definition* in any strict sense, because it does not permit the elimination of numerical terms in all contexts. This does not settle the issue, however, since it may be claimed that the principle is *analytic*, or a conceptual truth, in some sense broader than: either a truth of logic or reducible to one by means of definitions. That it can

[199] Hume's principle, in the language of second-order logic, is:
$\forall F\ \forall G\ (n(F) = n(G) \Leftrightarrow \exists R\ (\forall x\ ((Fx \rightarrow \exists! y\ (Gy \wedge xRy)) \wedge (Gx \rightarrow \exists! y\ (Fy \wedge yRx))))$

be so regarded is the view - now often called neo-Fregean logicism - of Crispin Wright and myself.[200]

(2) The logicist platonist claim: Frege was right both about the priority of syntactic categories to ontological ones, and about the syntax of arithmetic. For this reason, mathematics is about abstract objects which are objective, and mind-independent, and which belong to an ideal, non spatio-temporal located realm. Numerals therefore have referents, viz. numbers that are abstract, self-subsistent objects.

Of course, these claims are heavily programmatic. In particular, neo-logicism must vindicate the status of Hume's principle as a definitional, or at least conceptual, truth. And it must do so in such a way as to sustain Frege's conviction that, in so far as syntax is prior to ontology, mere logical and/or conceptual truths can have the ontological commitments which Frege took to reconcile logicism with platonism. In the next two sections, respectively, I will consider how neo-logicists have approached these two tasks.

2. Hume's principle

Having derived Hume's principle from the explicit definition he gave of natural number in purely logical terms (including Basic Law V), Frege himself was left in no doubt as to the status of Hume's principle: it is analytic, being grounded in pure logic via explicit definitions. But Neo-Fregeans[201] do not derive Hume's principle from any other principles, logical or otherwise. They are therefore obliged to maintain that it is in *some* sense analytic, albeit not in the sense of being derivable from pure logic by means of explicit definitions. In this re-

[200] Hale (2000), p. 102.

[201] See Hale and Wright (2001), pp. 2-13.

spect, the avenues open to them are very limited. They have no choice but to treat Hume's principle as an *implicit* definition. Unsurprisingly, the thought that implicit definitions must meet certain conditions if they are to be acceptable in general, and still more if they are to meet the requirements of the neo-logicist's ontological and epistemological requirements, has generated a certain amount of controversy amongst neo-logicists regarding the precise form Hume's principle should take.

Clearly, if it is to be acceptable, an implicit definition must be consistent, both in itself and in relation to any system to which one chooses to add it. But although I have spoken of the "consistency" of Hume's principle with second order logic, strictly speaking this is untrue in one sense. It all depends on what one means by "Hume's principle". To recall, Hume's principle states necessary and sufficient conditions for the identity of "the number of F's" and "the number of G's" – namely, one-to-one correspondence between the F's and the G's. But although I gave no indication of the fact, unless some restriction is placed on permissible substitutions for F and G, Hume's principle thus stated is hyper-inflationary. The universe it generates is not merely infinite, and, hence big enough to generate the Peano axioms. It is actually *so* big as to be contradictory. This problem is at the heart of some critics' contention that neo-logicism is something of a cheat, in that the mere fact that formal restrictions have to be placed on the scope of Hume's principle so as to preserve consistency, and some care has to be taken in choosing these restrictions, shows that Hume's principle cannot possibly have the status of a mere *definition*. However, I think this criticism is weak. Consistency is a requirement of *any* definition, implicit or otherwise, and no one should suppose that consistency in definitions or otherwise comes cheaply. Accordingly, unless implicit definition ruled out altogether *en masse*, there is no reason to object to the use of a restricted version of Hume's principle as an implicit definition of an operator "the number of ..". which forms a singular term out of a predicate.

In any case, neo-logicians have tended to advocate restrictions on Hume's principle which go far beyond the mere demands of consistency. The idea that (a restricted version of) Hume's principle might constitute an acceptable implicit definition of the operator "the number of ..." is naturally grounded not just in the thought that this much information suffices to confer sense on this operator, but in the further thought that this much information is at least partly constitutive of an understanding of the concepts of arithmetic, and, more precisely, of the concept of natural number. From this view point of view, an implicit definition by means of (the version of) Hume's principle (to be employed) is tacitly known by those who understand arithmetic. But various authors have argued only a quite *severely* restricted version of Hume's principle is partly constitutive of an understanding of arithmetic.

Heck[202], for example, asserts that without quite severe restrictions, Hume's principle is not implicit in an understanding of arithmetic. Heck is worried that Hume's principle goes beyond what is constitutive of an understanding of arithmetical concepts quite early on. In particular, he is worried even at the stage at which Hume's principle states identity criteria for the number of *infinite* F's to be the same as the number of *infinite* G's. Let us call a version of Hume's principle which permits the substitution of two infinite concepts F and G "infinite Hume's principle". Heck takes ordinary arithmetical reasoning to be 'reasoning about, and with, finite numbers', and his basic claim is that 'no amount of reflection on the nature of arithmetical thought could ever convince one of [infinite] Hume's principle'. We count by determining a one-to-

[202] See Heck (1997).

one correspondence between an initial sequence of numerals and the objects we want to count. We start with the numeral '1' and end with a numeral 'n' that stands for the number of objects. Accordingly, in so far as infinite Hume's principle says that equinumerous concepts have the same cardinality even in the case of infinite cardinals, Heck thinks it is *not* implicit in ordinary arithmetical reasoning. He suggests that nothing in the latter encompasses the idea of that one-to-one correspondence is a criterion for identity in case of infinite cardinals as well. He takes the example of the concepts *natural number* and *even number*. The even numbers are a subsequence of the natural numbers. It is by no means obvious that there are as many even as there are natural numbers. Given our naïve, intuitive conception that there are more natural than even numbers, and that a proper part is always smaller than the whole (collection, segment and the like) in the finite as well as in the infinite domain, to many it comes as something of a surprise to learn. Since it is therefore not something an ordinary arithmetical thinker would think of, it is not part of our ordinary arithmetical thinking.

The negative moral Heck draws from these reflections is that neo-logicism cannot appeal to infinite Hume's principle: that principle is too strong to constitute an implicit definition of natural number. The strongest principle neo-logicism is in a position to appeal to is one he calls Finite Hume's principle. Finite Hume's principle gives the conditions for two concepts, F and G, *at least one of which is finite*, to have the same cardinal number:

$$\forall F\ \forall G\ (\ (\mathrm{Finite}(F) \vee \mathrm{Finite}(G)\ ) \rightarrow\ (\ n(F) = n(G)\ \Leftrightarrow F \approx G\ )\ )$$

In contrast the infinite version, Finite Hume's principle 'really is implicit in arithmetical reasoning'. For 'one can convince oneself of its truth, come to understand why it is true, by (and perhaps only by) reflecting on basic aspects of arithmetical thought'. Finite Hume's principle does not require the conceptual leap that Hume's principle does, and almost all mathematicians (Heck mentions Bolzano) have accepted it. However, Heck does offer the neo-logicist a positive result. The neo-logicist account of arithmetic as such has no need of Infinite Hume's principle. In second order logic, Finite Hume's principle suffices for the derivation of Peano's axioms.

Prima facie, Heck's critique of Hume's principle is less a threat to neo-logicism, than a vindication of it. Upon further reflection, however, it is something of a double-edged sword. In arguing that neo-logicism can only appeal to conceptual principles that are implicit in ordinary arithmetical understanding, Heck makes it more vulnerable than it would otherwise be. He himself thinks Finite Hume's principle meets this criterion, but others hold that even this much goes too far. In particular, MacBride questions Heck's claims that Finite Hume's principle is implicit in ordinary mathematical practice, and can be arrived at by reflection on the process of counting.[203] According to MacBride, it cannot be acquired via a simple reflection on the process of counting, since it requires an underlying understanding of the notion of infinite cardinal. An example might be the concept F being the concept *natural number* and G being the concept *natural number whose square is identical to 9*; or any other example in which one of them, e.g. F, is infinite while the other - G is finite. We cannot make sense of the identity between two numbers one of which is an infinite cardinal number on the basis of our ordinary arithmetical understanding, since it cannot be grasped from our understanding of the process of counting. An infinite cardinal number belongs to a concept whose objects that fall under it cannot be counted; viz. the objects cannot be put in one-to-one correspondence with any initial segment of the sequence of numerals.

[203] MacBride (2000).

The only version of Hume's principle that arises from the counting process is, MacBride argues, one he calls Weak Hume's principle. It gives the condition for two *finite* concepts to have the same cardinal number:

$$\forall F \forall G \, ((\mathrm{Finite}(F) \wedge \mathrm{Finite}(G)) \rightarrow (n(F) = n(G) \Leftrightarrow F \approx G))$$

I suspect that Weak Hume's principle is strong enough for the neo-logicist's purposes: it too will suffice to generate the Peano axioms. After all, it says, in effect, that for every concept F under which finitely many objects fall, there is the number of F's. But in any case, I am by no means convinced even by Heck's claim that Infinite Hume's principle is not implicit in ordinary arithmetical understanding. The *unrestricted* criterion of one-to-one correspondence as a criterion of identity is actually something very close to the understanding even of children. The conceptual leap is not so much to the criterion for infinite cardinals, but the actual infinity itself. Once the actual infinity as such was accepted, and that was historically the main problem, the further supposition that the same criterion of identity to apply to infinite sets/collections as well was genuinely seemless.

Moreover, Heck is surely wrong to restrict neo-logicism to what is implicit in *everyone's* arithmetical understanding. I do not see why the ordinary arithmetical thinker should be a protagonist. Heck argues that Infinite Hume's principle cannot be implicit in ordinary arithmetical reasoning, because it takes mathematical genius – of the kind exhibited by Cantor - to formulate it. But so what? Isn't the actual history of mathematics the history of mathematical genius? Even if all what we want to know is how something is actually grasped, and choose to restrict ourselves to what is implicit in mathematical understanding, we should analyse the work not of the man in the street, but of the greatest mathematicians. To suppose otherwise is like protesting the laws the astronomy on the grounds that no ordinary person would ever have imagined the earth moving around the sun. And once one focuses on mathematicians, Infinite Hume's principle is there for the taking. As MacBride himself observes, it is not as if Cantor introduced an utterly novel principle of numerical identity. If pre-Cantorian mathematicians had endorsed just *Finite* Hume's principle, there would have been no possibility for them to even discuss the example of natural and even numbers, since Finite Hume's principle says nothing about identities amongst infinite numbers. But of course these matters had been discussed amongst mathematicians long before Cantor. MacBride rightly mentions Bolzano's *Paradoxes of the Infinite*[204], as just one possible example, and the "paradox" that there are as many even numbers as numbers was known e.g. to Galileo[205].

Finally, I think it a retrograde step in any case to focus on what is implicit in *anyone's* understanding. So doing flouts what we saw to be the first of Frege's methodological principles, namely, his anti-psychologism (see Chapter 7 above). Focusing on the psychological in this way is simply not what Frege, or his neo-Fregean followers, is about. As MacBride says:

> That project never was to uncover *a priori* truth in what we ordinarily think, but to demonstrate

[204] Bolzano (1950).

[205] See Galileo (1914).

> how *a priori* truth could flow from a logical reconstruction of arithmetical practice.[206]

The neo-Fregeans' aim is to give an explanation of how Peano axioms *can* be grounded in a version of Hume's principle (via second order logic), not how the axioms *are* grasped or what an average arithmetical thinker, whatever that means, has in mind when performing ordinary arithmetical reasoning.

On balance, then, Heck's critique, and in particular his derivation of the Peano axioms from Finite Hume's principle, provides the neo-Fregean program with significant support.

3. Abstraction principles: Theft over honest toil?

Following Frege himself, neo-Fregeans[207] maintain that it is possible to define abstract sortal concepts - that is concepts whose instances are abstract objects of a certain kind - by stipulation. What has to be stipulated is the truth of an abstraction principle.

Abstraction principles have the general form:

$$\forall f\, \forall g\, (\, \psi(f) = \psi(g) \leftrightarrow f \approx g\,)$$

Here, f and g are variables referring to entities of a certain kind (objects or concepts usually), ψ is an operator which forms singular terms when applied to f and g – so that $\psi(f)$ and $\psi(g)$ are singular terms referring to objects, and $\approx$ is an equivalence relation on entities denoted by f and g.

[206] MacBride (2000), p. 158.

[207] See, for example Hale (1999b).

Frege himself invokes three abstraction principles:

The direction principle:
The direction of the line *a* is identical to the direction of the line *b*
if and only if
the line *a* is parallel to the line *b*.

Hume's principle:
The number belonging to the concept F is identical to the number belonging to the concept G
if and only if
the concept F is equinumerous with the concept G.

Basic Law V:
The extension of the concept F is identical with the extension of the concept G
if and only if
every object that falls under F falls under G and conversely.[208]

The third one notoriously turns out to be inconsistent.

Abstraction principles are of major importance for neo-logicism. They bear the main burden of the task of reconciling logicist or neo-logicist thesis that arithmetic and analysis are pure logic. In so far as they are *stipulations* they can aspire to explain in one stroke both how *logic* can be committed to abstract objects, and how it is possible to have knowledge of these objects. For, in typical cases – such as the direction principle – our knowledge of instances of the right hand sides of the equivalences is relatively unproblematic. We know on occasion that one line is parallel to another. If the corresponding abstraction principle is to be believed, then, a mere stipulation affords us knowledge on this basis of something else – namely, that one abstract object – the direction of the one line – is identical to another – the direction of the other.

[208] Frege formulates the first two in *Grundlagen*, while the third one in *Grundgesetze*.

On the other hand, the burden neo-logicists place on abstraction principles, following Frege, might seem unbearable. How can mere stipulation ensure the existence of the objects apparently referred to on the left hand side of an abstraction principle's equivalence sign, when nothing on the right hand side makes even apparent reference to such objects? How can objects the existence of which is independent of us and our practices be *stipulated* into existence? The neo-logicist answers that the stipulation guarantees – or shows, even – that the relevant abstract objects's existence *consists* in the truth instances of the other (right-hand) sides of such principles. As Hale points out in the case of the direction principle:

> directions simply are (on this explanation) objects for whose identity (and therefore for whose existence) it is necessary and sufficient that the corresponding statement of line-parallelism be true.[209]

Fundamental to this conception of the (possible) role of abstraction principles is the idea that the objects on the left hand side of an abstraction principle's equivalence sign introduce no further *content* than is already presupposed on the right hand side. So to speak, for example, if it is given that line *b* is parallel to line *c*, then the idea that an object – the direction of *b* – is identical to another – the direction of *c* – is given too. This idea gives *the same fact*. In Hale's terminology, the right and the left-hand sides of abstraction principles are simply "carving up" one and the same content in a different way.

[209] Hale (1999b), p. 94.

The Fregean perspective on abstraction principles promises to be, then, something of a holy grail of platonism. Our knowledge that one line is parallel to another is not in doubt. But now we are told that, in effect, this knowledge *is* knowledge of abstract objects. Our knowledge that there is a one-to-one correspondence between the F's and the G's is not in doubt. But now we know that, in effect, this knowledge *is* knowledge of numbers – and, in particular, of the number of F's – namely, that this number is identical to the number of G's. Unsurprisingly, however, the Fregean perspective is not unproblematic. The crucial notions are that of content, and of "recarving".

We need to examine them carefully.

As far as content is concerned, the crucial question is this: How is content, and in particular sameness of content, to be defined? This question is the focus of a recent dispute between Hale and Potter/Smiley, and I will look at this dispute in some detail[210].

According to Potter[211], the fact that Basic Law V is also an abstraction principle prevents a suitable notion of content to be defined in such a way as to give the left and the right sides of, respectively, Hume's principle and the direction principle, the same content. Sameness of content cannot but depend on the syntactic structure of the principle alone. But Basic Law V shares this structure too. So if it were possible to define a suitable notion of content so that the left hand side in an abstraction principle recarves the content of its right hand side, the same could be said for Basic Law V. But in that case the principle would be true. Since it is inconsistent, such a definition of content is not possible.

[210] See Potter (1999), Hale (1999b), Potter and Smiley (2001) and Hale (2001).

[211] Potter (1999), p. 67.

Hale's response to this criticism is to deny that the syntactic similarity Basic Law V and other inconsistent abstraction principles bear to principles such as Hume's or the direction principle does not preclude a criterion whereby the former are unacceptable and the latter are acceptable. This "bad company argument" is flawed. Its moral is simply that the desired criterion of sameness of content is not just a syntactic matter. Secondly, and more importantly, in an attempt to substantiate this attitude, Hale offers a criterion for sameness of content which he claims meets the neo-logicist's requirements. The criterion stems from the observation that

> Anyone who understands a statement of direction-identity via the stipulation of the Direction Equivalence can tell, without inference, that it must have the same truth-value as the corresponding statement of parallelism.[212]

Hale's criterion of sameness of content gives a twist to the hint this observation offers:

> Two sentences have the same truth-condition (content) iff anyone who understands both of them can tell, without determining their truth-values individually, and by reasoning involving only *compact* entailments, that they have the same truth-value.[213](my emphasis)

Here, the crucial notion is that of "compact" entailment. This is defined as follows: $A_1, \dots, A_n$ compactly entail B if and only if

(i) $A_1, \dots, A_n$ entail B, and

[212] Hale (1999b), p. 98.

[213] Hale (1999b), p. 97.

(ii) for any, that is for all non-logical constituent E occurring in $A_1, \ldots, A_n$, there is some substitution E'/E which applied uniformly through $A_1, \ldots, A_n$, yields substitution instances $A'_1, \ldots, A'_n$ that do not entail B.

The intuitive idea, then, is that an entailment is compact when some uniform substitution in the antecedent to the entailment undermines the entailment.

The condition that reasoning has to involve *compact* entailment is weighty since it rules out the counterintuitive eventuality that necessary truths have the same content. For any two necessary truths A, B the condition (i) in the above definition is fulfilled: A entails B and vice-versa. Hence, in the absence of the further condition (ii), A and B would have the same content. As Hale points out, by defining the compact entailment as a condition for sameness of content, it turns out that the abstraction principles' left and right hand side do recarve the same content.

Potter and Smiley[214] are unconvinced. They find Hale's criterion of sameness of content flawed in detail, and they continue to find his neo-logicist perspective on abstraction principles misguided in principle. Their objection that Hale's notion of compact entailment cannot provide a 'credible explication of "content" amounts to three claims.

First of all, Hale's criterion fails the basic requirement of any criterion of sameness of content. As they put it, 'the most urgent question, ... , is whether Hale's criterion for identity of content succeeds in setting up an equivalence relation, since otherwise, ... , it cannot be a criterion of identity for anything'.[215] But it does not succeed in this! The implication $A \rightarrow A$ does not compactly entail itself since: $A' \rightarrow A'$ entails $A \rightarrow A$ (or in symbols: $A' \rightarrow A' \vdash A \rightarrow A$). Hence, mutual compact entailment is not an equivalence relation. Hence, sameness of content by Hale's criterion is not an equivalence relation. Hale's criterion therefore has the absurd consequence that $A \rightarrow A$ does not have the same content as itself. Moreover, taking every substitution instance of a compact entailment as compact too in an attempt to solve the problem, will not work; for a new problem arises; namely, the loss of transitivity.[216] This, of course, is no less disastrous than a failure of reflexivity. A relation is not an equivalence relation unless it is transitive.

[214] See Potter and Smiley (2001).

[215] Potter and Smiley (2001), p.331.

Apart from these difficulties of detail, Potter and Smiley find two difficulties of principle in Hale's treatment. The first reiterates Potter's original complaint regarding Basic Law V. They observe that 'if the idea of content and its recarving is to be more that a vague metaphor, it must succeed in discriminating the good from the bad [abstraction] principles'[217], where a good principle is the one whose content of the left-hand side can be carved up in a new way from the content of the right-hand side:

> ... the proposal must be that an abstraction principle is good if and only if the content of its right hand side is recarvable to form the content of its left hand side. For the point of introducing the idea lies precisely in the contention that an abstraction principle, once enunciated, can be seen as analytic of the abstraction it introduces, and hence as requiring no further justification.[218]

Their view remains, however, that this just cannot be done. Content has to 'belong to the realm of understanding rather than semantic value, i.e. it must be, roughly speaking, on the

[216] Potter and Smiley (2001), p.329.

[217] Potter and Smiley (2001), p.332.

[218] Potter and Smiley (2001), p.332

side of Fregean sense rather than reference'[219]. The idea underlying this claim is as follows. In Hume's principle (or **Num**)[220] we do not proceed or start from the fact that numerals refer to numbers: that is the goal wė are supposed to achieve with the principle. Since we do not have an account of how numerals refer to numbers prior to **Num**, it is necessary to have an explanation of content independently of the semantic value of the terms that occur in it. Clearly, a notion of content of this kind lies in the domain of understanding (or Fregean sense), rather than of reference. But this is to say that it is insensitive to the bad abstraction principles that produce too many objects. The right hand sides of the bad abstraction principles make perfectly good *sense*; if they did not we would have not been seduced into naïve set-theory.

In order to explain in more detail where the difficulty lies, Potter and Smiley divide the task of an abstraction principle into two steps. The first step is to recarve an appropriate relation into an identity between concepts, and the second one is to transform it into an identity between objects. A relation is suitable if it is an equivalence relation. In the case of **Num** we have: $(\forall F)(\forall G)$(the concept '$\sim F$' = the concept '$\sim G$' $\leftrightarrow$ $F \sim G$)[221]. The second step – which could turn out to be impossible – is about associating an object with each concept that appears on the left-hand side of the equivalence. Given initially k individuals, there are 2^k extensionally different concepts of individuals represented in the above formula by the Fs and the Gs, which are grouped by the relation '$\sim$' under a number of second-level concepts of the form '$\sim F$'. If there are more than k of them, associating them with distinct individuals cause the domain of individuals 'to explode in a paradox of impredica-

[219] Potter and Smiley (2001), p.333.

[220] Potter and Smiley call it the number principle or briefly **Num**, so I shall refer to it as to **Num** too.

[221] The notation '$\sim F$' means 'being equinumerous with F'.

tivity'[222]. Given the general notion of content, the content of the one-to-one correspondence does not and could not embrace the condition of not having more than k concepts of the form '$\sim F$'. It would be like 'claiming that it is part of the content of 'authors of *Principia*' that there should be no more than two of them'[223].

Potter and Smiley's second objection of principle is perhaps even more fundamental. In their view there is a principled reason why abstraction principles cannot produce objects: 'objects cannot be conjured into existence by stipulation-nor by definition, reconceptualising, recarving of content or any other move within the realm of sense'[224]. Hale's contrary belief is based on his assumption that singular terms cannot lack a reference. But Potter and Smiley hold this assumption to be erroneous. Singular terms in true statements *can* lack a reference. They argue that to appreciate this fact it is necessary to distinguish two possible readings of identity. There is a strong reading, the one Hale takes for granted in his formulations of abstraction principles, according to which:

$a = b$ is false if either a or b or both fail to refer.

But there is also a weak reading though, according to which:

$a = b$ is false just if either a or b fail to refer, but it is true - vacuously true - if both terms fail to refer; the weak reading can be defined in symbols: $(\forall x)$ ($x=a \leftrightarrow x=b$).

Since identity implies existence on the strong reading, but not on the weak one, Hale's neo-logicist employment of abstraction principles begs the question. It presupposes that the principles can be stipulated to be true with identity read in the *strong* sense. In reality, however, the most that could be

[222] Potter and Smiley (2001), p.334.

[223] Potter and Smiley (2001), p.334.

[224] Potter and Smiley (2001), p. 337.

said of some such principle is that it is stipulatively true provided identity is used in the *weak* sense.

In Potter and Smiley's view, then, Hume's principle can be divided into two separate principles, which they term **Pure Num** and **∃! Num**.

Pure Num has no commitment to the existence of numbers. It amounts to:

Pure Num

$(\forall F)(\forall G)(\,(NxFx = NxGx \rightarrow F{\sim}G) \wedge (F{\sim}G \rightarrow NxFx \equiv NxGx)\,);$

The second one, which explicitly asserts the existence of exactly one number that belongs to every concept, amounts to:

∃! Num $(\forall F)(\exists!\, NxFx)$

As they summarise their objection, 'the most Hale is entitled to is the existentially non-committal **Pure Num**'.

Hale's reply to Potter and Smiley's criticism is puzzling.[225] He maintains that their Smiley complaints rest upon serious misunderstandings.

Hale's response to Potter and Smiley's criticism of his specific criterion of sameness of content goes as follows. He admits that the charge that compact entailment is not reflexive and/or transitive is true. However, he protests that this fact does not represent a problem since the proposal was not to define content identity directly as compact equivalence. Indeed, he insists that in certain cases, appreciation of identity of content requires no reasoning at all. In particular, no reasoning is required to appreciate that some simple sentence must be alike in truth value with itself. In effect, this is to admit that his original criterion was poorly expressed. It should have stated that A and B have the same content if and only if understanding both suffices for an appreciation that they

[225] See Hale (2001).

have the same truth value, prior to determining which truth value they have, either by non-inferential means or else by inferential means confined to compact entailments. But even then Hale's response is problematic. Can one recognise that a sentence A has the same truth value as A without reasoning? Given a sentence such as, for example, "2+2=4", do we not (at least implicitly) think something like: "For every sentence A: A has the same truth value as A, therefore "2+2=4" has the same truth value as itself".

Hale's response in this regard is rather at odds with the fact that, even so, he chooses to amend his criterion in a more substantial respect. For he proposes to revise the definition of compact entailment as follows: *A* compactly entails B if and only if

(i) *A* entails B

(ii) for every non-logical expression E in A, there is some E′ such that A(E′/E) - that is, the result of substituting E′ for E uniformly throughout A - does not entail B

(iii) for every subformula S of A, there is some modal equivalent S′ such that A(S′/S) does not entail B.

Regarding their objections of principle, Hale complains that they are due to a misinterpretation and overestimation of the neo-Fregean project. It is true that an abstraction principle is good if and only if the content of its left-hand side is recarvable from the content of its right-hand side. However, it is not part of the project that an abstraction principle can be seen as analytic with no required further justification. Neo-Fregeans maintain that the stipulation of the truth of an abstraction principle is acceptable only under certain constraints. As Hale points out:

> These constraints include, among others, requirements of consistency, and more generally, a certain kind of conservativeness which demands - to put it somewhat loosely - that the

> stipulation should not have the effect of settling the truth-values of statements, comprising only vocabulary which is already in place, whose truth or falsity ought to be a matter of independently constituted fact.[226]

If all of the constraints are not fulfilled - and this could happen without us being aware of it - the (attempted) stipulation simply fails. All the constraints have therefore to be satisfied in order for an abstraction principle to be a good one. The fulfilment of such constraints is the criterion for the goodness of an abstraction principle. The criterion for a principle to be a good one that Potter and Smiley accept is therefore not the criterion itself but just a consequence of the fulfilment of the above mentioned constraints. Accordingly, the problem of Basic Law V is easily solved; as is well known, it does not meet such conditions.

Finally, Potter and Smiley's protest that abstraction principles cannot stipulate objects into existence is again a misunderstanding. Neo-Fregeanism has never claimed that this is what abstraction principles do. Laying down an abstraction principle is *not* about creating objects. It is about introducing a concept which independently existing objects fall under:

> Quite generally, what - always provided that the appropriate constrains are met - the stipulation of an abstraction principle 'produces' is not objects, but a concept.[227]

The 'produced' concept can, on the other hand, either be instantiated or not: it depends on the truth or falsity of the right-hand instances which is certainly *not* a matter of stipulation.

[226] Hale (2001), p. 345.
[227] Hale (2001), p. 347.

10

Neo-Logicism (2) – The Solution to Platonism's Difficulties?

We have seen in the previous chapter how contemporary neo-logicism attempts to preserve what it considers to have been the insights of Frege's logicism, while at the same time avoiding the pitfalls of his theory of extensions. Having considered both the formal and the philosophical development of the neo-logicist program, we are now in a position to evaluate it. This is the aim of the current chapter. In particular, we need to consider whether, in essence, Frege's treatment of the foundations of arithmetic and analysis affords solutions to the indeterminacy and epistemological problems facing platonism which neo-logicism can exploit.

I begin with a problem of which neo-logicists are all too aware, and which I have yet to address. This is the "Caesar problem", the problem Frege himself raised for what was, in effect, his anticipation of neo-logicism, and which he himself took to undermine the neo-logicist project.

1. The Caesar problem

To recall, the problem Frege raises for the proposal to introduce the concept of number via the identity criterion provided by Hume's principle, is this: prima facie, this criterion falls short of what is required, because it fails to give sense to identity-statements which are not of the form Hume's principle addresses (namely, "the number of F's is identical to the number of G's"). In particular, they fail to determine the sense of mixed identities like "The number of F's is Julius Caesar". By implication, Frege himself takes this problem to be insoluble, and resorts to an explicit definition of natural number from which Hume's principle can be derived. This treatment bypassed the Caesar problem. According to Frege's definition, natural numbers are the extensions of higher-order concepts of the form "equinumerous with the concept F". Since these extensions comprise lower-order concepts, and Caesar is not a concept, the sense, and truth value of a statement "the number of F's is Julius Caesar" is determined. In the explicit definition of an object, its criterion of identity is given as comprehensively as possible.

Neo-Fregeans resist Frege's treatment of this problem. They resist his claim that introducing the natural numbers by means of Hume's principle leaves their criteria of identity undetermined in a way made explicit in the Caesar problem. In effect, they do so on the grounds that he misconceives the nature of so-called "sortal concepts". The key to their proposal is the notion of a "sortal concept", and in particular, the idea that sortal concepts themselves fall into different categories.[228]

What is a sortal concept? A concept F is sortal iff it always makes sense to ask how many F's are there that satisfy a certain condition (given that the stated condition itself makes sense). So, clearly, the concepts cat, person, book and so on are sortal. By contrast, to give Frege's own example, the concept 'red' appears not to be. No doubt in certain contexts one might sensibly ask "How many red things are there?". But one cannot do this in general. This is because the answer cannot be determinate unless context supplements the question with some further concept – 'book', 'pen' etc. – which is itself sortal. In the absence of such supplementation, the question appears not to have a determinate answer. To be sure, if, after an accident, one's faculties were being tested, and, confronted with a table on which seven figurines were placed, three of

[228] See, for example, Hale and Wright (2001), pp. 336-396.

them red, one was asked "How many red things are there?", one might immediately answer "There are three of them". But that is just to say that one understood the question to be "How many figurines on the table are red?" Had one not provided a supplementary sortal concept in this way, one would have not known what to answer. Here is one red figurine. But it has two arms which are red, and two legs too; so does that make four red things? No, for the arms have hands, and they are red too etc. The concept of natural number is paradigmatically sortal: numbers are used to count, but they can also be counted. As Hale and Wright say, when Frege says that numbers are objects, 'he is best taken to mean that Number is a (non-empty) sortal concept in this technical sense'.[229]

It is clear from their definition that sortal concepts involve a criterion of individuation: Are A and B one book or two books?; and a criterion of identity over time: Is A the same book I bought yesterday? These criteria are essential to the concepts. For example, unless books are determinately individuated, the question "How many books are on the table?" would have no answer. Accordingly, the concept of natural number must likewise embody criterion of individuation. Unless whether two numbers were identical were always a determinate matter, it would not be the case that the question "How many numbers are there?" always makes sense.

This much does little more than to sharpen our understanding of the Caesar problem. In effect, it merely allows a restatement of Frege's claim that the concept of natural number embodies a criterion of individuation which Hume's principle alone is unable to provide. However, we are now in a position to understand the neo-logicist rebuttal of Frege's attitude.

[229] Hale and Wright (2001), p. 367.

As I said, this turns on the notion of "categories" of sortal concepts. A category 'may usefully and naturally be identified with a *maximally extensive sortal*'[230]. A maximally extensive sortal F is a sortal concept such that all the sub-sortals have the same criterion of identity and no other concept can be added as sharing the same criterion of individuation. An example might be the category, that is, the maximally extensive sortal concept, of "being a spatio-temporal located object". The criterion of individuation that works for all spatio-temporal located objects *might* be:

given two spatio-temporally located objects S_1, S_2:
$S_1 = S_2$
if and only if
at all times, S_1 occupies the same place as S_2.[231]

How does the notion of a maximally extensive sortal solve the Caesar problem? The crucial consequence of such a distinction is that no concept can belong to two different categories. So, inside of a category, objects can be differentiated by referring to the criterion of identity while objects that do not belong to the same category are differentiated precisely by the fact that they belong to different categories. So, if we take an object to which the criterion of individuation associated with the concepts of a category C could not apply, we have a guarantee that it does not fall under any of those concepts. It follows that Hume's principle alone guarantees that Julius Caesar is not a number. Since, Julius Caesar falls under the

[230] Hale and Wright (2001), p. 389.

[231] This criterion is nevertheless contentious. Wiggins for example would not agree with it since he thinks it is dodge to have examples like this: "This lump of clay is identical to this statue" even though clay and statue occupy the same place at all times. Wiggins holds that the lump of clay is not identical to the statue; it just constitutes the statue. But I have appealed to this criterion of individuation only to give an example. It is not my concern to defend it.

concept "person" (or Person), the criterion of individuation that applies to him is the criterion embodied in the concept Person and all the other concepts in the category to which the concept person belongs (whatever that is). In contrast, Hume's principle tells us that the concept "natural number" belongs to a different category: for its criterion of individuation, as provided by that principle – in terms of one-to-one correspondence between the entities which fall under a concept – is not a criterion which applies to people. Since Caesar is not subject to the criterion of individuation to which natural numbers are subjected (by Hume's principle), and vice-versa, it follows that Caesar cannot fall under the concept natural number. Hence, contrary to Frege's fears Hume's principle does tell us, after all, that Caesar is not a number. [232]

Perhaps it might be objected that such a solution to the Caesar problem requires a developed underlying theory that characterises or defines the concepts or the objects in question *prior* to applying Hume's principle, and that the need for such a theory was precisely what Frege was complaining about. However, this criticism misses the point. According to neo-Fregeans, Hume's principle does not merely offer a crite-

[232] Hale introduces a further distinction of sortal concepts, viz. pure, phase and impure sortal concepts. Pure sortal concepts are those I have characterised above as sortal and when he talks of sortal concepts he has in mind just the pure sortal ones. Phase sortal concepts, as Wiggins calls them, and Hale follows suit, are those like caterpillar, sapling, tadpole, and so on, with a transition from one phase to another. Impure sortal concepts are those formed by restriction of a pure sortal by 'some further inessential characteristic - a man with a ice screen, brown cow, river longer than 1000 miles, tiger with a thorn in his foot, and so on. I do not see the need for such a distinction though. Why would a tadpole represent a phase while a person does not? A tadpole will become a frog and a person will become a dead body. Why is "sapling" a phase sortal concept while "tree" is not? Just because the phase in the former case is shorter than the phase in the latter one? And what is the importance of those 'inessential characteristics'? If we ask: How many women with green eyes are there?, I think there is no philosophically important way in which such question is different from the question: How many women are there?

rion of identity for the (sortal) concept of number, or merely *partially* explains the concept:

> The neo-Fregean, however, makes a stronger claim - that by stipulating that the number of *F*s is the same as the number of *G*s just in the case the *F*s are one-one correlated with the *G*s, we can set up *number* as a sortal concept, i.e. that Hume's Principle *suffices* to explain the concept of *number* as a sortal concept.[233]

It is of the essence of neo-Fregeanism that Hume's principle alone suffices to give a *complete* explanation of the sortal concept "natural number", and, eventually, once suitable truths involving numerical singular terms have been proven, of the objects which fall under it.

I agree with neo-logicism in its dispute with Frege regarding the Caesar problem. The criterion of identity invoked in Hume's principle does settle the question as to whether Caesar is a natural number in the negative.

2. Neo-logicism's platonism

To say agree that Hume's principle establishes that Caesar is not a natural number (given that Caesar is a person), is to agree that it establishes that no identity statement of the form "Julius Caesar is identical to the number of F's" is true. And, hence, on the further assumption that every number is such that for some F it is the number of F's, it is to agree that no identity statement of the form "Julius Caesar is n", where n is any numerical singular term, is true. However, this concession falls far short of what neo-logicists require of Hume's

[233] Hale and Wright (2001), p. 15.

principle. For they take it, in combination with uncontentious facts, to generate *true* identity statements of the form "the number of F's is identical to the number of G's". And they take it to guarantee that the singular terms these identities involve refer, and, hence, that existential generalisation applies to them. In particular, one can infer from the truth of an identity of the kind just quoted that there is some x, such that x is identical to the number of G's. But to concede that the Caesar problem can be overcome is not to concede this much.

It is very easy to be seduced into thinking that the neo-logicist's claims for Hume's principle are legitimate. We all unthinkingly believe that (a suitably restricted version of) Hume' Principle is true. Hence, once we learn that the knives are one-to-one correlated with the forks, we automatically infer that then number of knives is identical to the number of forks. But of course neo-logicists construe the proposition we infer in this way differently from what we might suppose, and once their suppositions are made explicit, our readiness to make the inference might diminish. They read into our unthinking "the number of knives is identical to the number of forks" the further proposition that there *exists* an object – an abstract object no less – such that this object is identical to the number of knives. Of course, they argue that this further existential proposition is, upon reflection, a commitment which can be teased out of the numerical identity we unthinkingly endorse. But the fact remains that once it is made explicit, and accept it, we are liable to be less sure than we were that the one-to-one correlation between knives and forks itself suffices for the truth of that identity.

The reader might accuse me of inconsistency at this point. In Part 1 I argued that platonism is true. Isn't my present reluctance to accept the neo-Fregean attitude to abstraction principles in general, and to Hume's principle in particular, inconsistent with that position? Aren't I simply expressing doubts about platonism which I resisted earlier? Well, no, I am not. My position remains firmly platonist. It is not platonism that I am currently resisting, but the neo-logicist conception of the *ground* of platonism. When I argued for platonism in Part 1 I did not argue that that numbers exist, and are abstract objects, is stipulatively implicit, as a matter of sheer logic, in facts like: the knives are one-to-one correlated with the forks. On the contrary, I emphasised the indispensability of mathematics to physical science (in addition to the "obviousness" of certain mathematical truths) as the ground of realism about truth value (in the strong sense), and, hence, for realism about ontology (and hence platonism). There was no commitment in any of that to the neo-logicist's conception of the conceptual and epistemological connection between number theory and Hume's principle; quite the opposite.

Ultimately, I think, the neo-logicist employment of Hume's principle boils down to Frege's thesis that syntax is prior to, and the arbiter of, ontology. It is hard – though perhaps not impossible – to maintain that a one-to-one correlation between the knives and the forks does not *itself* suffice for the *truth* of the identity: the number of knives is identical to the number of forks. What more could suffice for it? What more, so to speak, does the world have to do in order to make it the case that the number of knives is identical to the number of forks? Nevertheless, it is an extra step from the truth of this identity to the *objective* truth of this identity in the *strong* sense[234], and, hence to the existence of numbers. Following Frege's priority of syntax thesis, neo-Fregeans see this as no step at all. According to them, the required condition for singular terms to refer is that they

[234] See Chapter 2.

> occur in true statements free of all epistemic, modal, quotational, and other forms of vocabulary standardly recognized to compromise straightforward referential function. For if certain expressions function as singular terms in various true extensional contexts, there can be no further question but that those expressions have reference, and, since they are singular terms, refer to objects.[235]

A proposition that contains singular terms actually cannot be true unless those singular terms do refer. And when they refer, they refer to objects. In the case of mathematical expressions containing numerical singular terms, that is numerals, such expressions cannot be true unless there exist objects – numbers – to which the included numerals refer to. But in my terms this is question-begging. It is true enough for strong truth, for truth in the objective sense. But it is incorrect for truth in the weaker sense. Consequently, even if we agreed that Hume's principle is stipulatively true, the crucial question would arise as to whether stipulated truth is objective truth in the strong sense. Neo-Fregeans are simply wrong to assert that this question does not arise. In effect, this is a variant of Potter and Smiley's point at the end of the previous chapter. To settle that the statement "The number of knives is identical to the number of forks" is true, by stipulative reference to a one-to-one correlation between the knives and the forks, is *not* to establish that the numerical singular terms it involves refer. The truth of the identity "Hamlet is identical to Hamlet" tells us that much. Failure to recognise this fact lands Fregeans with a problem of circularity. They offer criteria for determining when an expression functions as a singu-

[235] Hale and Wright (2001), p. 8.

lar term. But by so doing they undermine the criteria we normally employ for ascertaining that the criterion is met. For example, the effect of the supposition that a one-to-one correlation between the knives and the forks *suffices* for the truth of the identity "the number of knives is identical to the number of forks" to be true, and, hence, for the existence of an abstract object to which the singular term "the number of knives" refers, is to make me wary of my earlier belief that the knives *are* one-to-one correlated with the forks. After all, it looks now as if in order to determine whether that much is true I have to ascertain whether the term "the number of knives" refers. Nothing has been said to prevent that from being a real issue, since nothing has been said to preclude the possibility that, contrary to my pre-reflective belief, it is not the case that the knives are one-to-one correlated with the forks. Moreover, I am left in an inescapable quandary. I can no longer be confident as to whether the condition for "the number of knives" having a referent is met, until I first ascertain whether "the number of knives" has a referent. The problem can also be illustrated by identity statements in general. According to Frege, an object exists if statements about it are objectively true; '*par excellence* statements of identity'. But how can an identity statements like '0=0' be distinguished from identity statements like 'Pegasus=Pegasus'? It seems that, no matter what we insert for '*t*', the identity statement '*t=t*' is always true. One answer might be that 'Pegasus=Pegasus' is not objectively true because 'Pegasus' is not referential. But the point is that we are supposed to know that 'Pegasus=Pegasus' is not objectively true prior to our knowing that 'Pegasus' is not referential. We are supposed to distinguished identity '*t=t*' in which '*t*' is referential from those in which '*t*' is not referential *before* we acknowledge if the term '*t*' is referential or not. This problem, I would say, seems insoluble.

3. Platonism's epistemology

Perhaps it is inevitable that neo-logicism's appeal to abstraction principles should be dismissed with a heavy heart. That appeal, after all, offered such welcome epistemological benefits. However, I do not think platonists should despair. A satisfactory epistemology for platonism is already evident in the considerations on which the attractions of platonism largely rest. It was noted in Chapter 5 that the claim that the problem of mathematics' applicability to the empirical world is particularly acute for platonism is somewhat ironic, since it is the fact of mathematics' applicability to the natural world which provides one of the main grounds for the realism about mathematics which platonism exemplifies. But at this point the dialectic comes full circle. For I want to suggest that the efforts of structuralists and neo-fregeans notwithstanding, it is the indispensability of mathematics to natural science – and hence its applicability – that provides platonism with the epistemology it needs, and, hence, which makes mathematical knowledge possible.

Consider again the very simple illustration of the problem of applicability as it appeared above in Chapter 5. I argued that 2F's + 2F's = 4F's is true in virtue of the fact that 2 + 2 = 4. That is the ontological order of things. But what about the epistemological order of things? One might think that it is the same. 2Fs+2Fs amounts to F+F+F+F, and this is like saying 1+1+1+1, given that we treat F as a sort of unit. And then the fact that the latter sum is 4 informs us the former sum is 4F's. We are able to abstract details and treat, for example, cats, as units. We know that sequences of concrete objects can exemplify mathematical structures which permit us to conclude that it is possible to (sort of) apply number equations to equations with concrete objects like 2Fs+2Fs = 4 F's.

However, Russell expresses a very different viewpoint when he writes:

> The proposition 2 + 2 = 4 itself strikes us now as obvious; and if we were asked to prove that 2 sheep + 2 sheep = 4 sheep, we should be inclined to deduce it from 2 + 2 = 4. But the proposition '2 sheep + 2 sheep = 4 sheep' was probably known to shepherds thousands of years before the proposition 2 + 2 = 4 was discovered; and when 2 + 2 = 4 was first discovered, it was probably inferred from the case of sheep and other concrete cases.[236]

Since we now find 2+2 = 4 obvious, and deduce that 2sheep+2sheep=4sheep from it, what are the grounds for Russell's confidence that the latter was been known to us before the former? Couldn't the shepherds have found it obvious that 2 + 2 = 4 is true, and then applied this knowledge in order to count sheep?

Well, perhaps. But the indispensability argument suggests that at least in more complicated cases, and perhaps even in this one, the basic thrust of Russell's position is correct: the epistemological order of things is the reverse of the ontological one. It is the applicability of mathematics, and in particular its indispensability to science, which gives us most reason to think that the mathematical theorems we take to be true really are true, independently of us and of our mathematical practice. Since mathematical knowledge amounts to reasonable, grounded, true belief, and this fact about the role of mathematics in our understanding of the natural world is our primary ground for mathematical belief, it follows that mathematical knowledge is grounded in this fact too. Obviousness might still have something to do with mathematical knowledge, ultimately. But it is the epistemological holism embodied in the argument from indispensability which really vindicates the platonist's conception of mathematical truth, and, hence, knowledge.

[236] Russelll (1907), p. 272.

4. The problem of indeterminacy

I think the neo-Fregean discussion of abstraction principles does make possible a solution to the problem of indeterminacy. It runs as follows. We might say that, since sets are all objects that there are, in the sense that mathematics is reducible to set theory, numbers are just a way of cutting the domain of sets. In obtaining numbers, that is pointing out to certain properties of certain sets, we ignore the rest of them (both properties and sets). In this way we create the domain of numbers such that two (or more) different cuttings of the domain of sets are possible, with different ways of cutting the domains of sets useful for different purposes. We point to certain properties of certain sets and give them a new name: numbers. When forming the domain of Zermelo's ordinals we denote such sets with numerals: 1, 2, 3, ..., and for them certain axioms (more precisely the Peano's axioms) are fulfilled. If we take the von Neumann's ordinals we notice that, by denoting such sets with the same numerals, the same axioms are fulfilled too. We therefore see that Zermelo's and von Neumann's sets have in common certain properties. Then if we reduce them to something we called numbers, we facilitate certain operations that are of great practical importance for us. In this way the problem of indeterminacy does actually not arise. Although sets objectively and independently exist, numbers are our creation, whose "existence" has no ontological commitment. Talking about numbers (and their mind-dependent existence) is shorthand for talking about certain sets (and their mind-independent existence).

I shall explain this view in more detail. I will firstly define what constraints such a solution has to meet, then what it exactly amounts to and finally shortly present Quine's solution that is in many respect similar to my own.

My constrains on a solution

Which constraints such a solution has to meet? Since it is possible to reduce arithmetic to set theory, that is numbers to sets, we have no ontological motivation for assuming the objective existence of numbers.

The development of mathematics (and in particular arithmetic) within set theory, justifies the application of the canons of economy and explanatory unification, that is the use of the celebrated principle of Ockham. The conclusion to be drawn is that sets, unlike numbers, are all that objectively exist. By concluding differently, that is by insisting to hold both numbers and sets in the objectively existing mathematical domain, we would get an (unnecessarily) overcrowded ontology - without any eligible consequence and with the problem of indeterminacy that would make such a thesis frail. We are therefore justified in asserting that all we have to be ontologically committed to is set theory, that is sets. Arithmetical statements are, if construed at face value, statements about (certain) sets, were different possibilities are viable. A solution of the indeterminacy problem must therefore show how a non-structuralist platonistic view of number-theory can accommodate the just mentioned constraints, viz.:

(a) it is not the case that numbers are abstract objects other than sets

(b) it is not the case that just e.g. Zermelo gives the correct account of what pre set-theoretic number theorists were referring to

My solution and the comparison of my view with Quine's

But an even more puzzling problem arises: did the e.g. 18th century mathematicians not refer to something else when they were talking about numbers than we do today when talking about the same(?) numbers? The answer seems to be affirmative. Namely, 18th century mathematical statements were about numbers, as *bona fide* objects. Number-theoretic statements should be hence differently conceived prior and after the discovery of the non-objective existence of numbers and their reductions to sets. Prior to such discovery, e.g. in the 18th century mathematical statements were, taken at face value, false. When Marie Antoniette literally lost her head, mathematics was literally false; because of the (wrong) assumption that truth makers for arithmetical statements were numbers. Numerals referred to numbers and the domain of numbers, construed as objectively existing objects, was taken to be the truth maker for the arithmetical statements. According to 18th century mathematicians numerals did refer to abstract, (as we now know) non-existing objects - numbers. The truth conditions of e.g. "1+2=3" in the 18th century was thought to be the following one:

"1+2=3" is true iff there exist the numer 1, the number 2 and, given the operation '+', the result of this operation upon these two numbers is the (existing) number 3.

In the 18th century therefore, mathematics - as construed as it was at the time - was false, since the assumption that numbers were the truth makers of arithmetical statements was false. Mathematicians were justified to believe mathematics was true because of its applicability and allegedly indispensability. The question that at this point appear is the following one: why should we accept the conclusions of the 18th century mathematicians now, given that they were wrong? And how come was the 18th century mathematics applicable givent that it was false? Simply because (past) arithmetical statements can be interpreted into set-theoretical statements which are objectively true and every arithmetical statement that was meant to be true correspond to objectively true statements concerning sets. The applicability part is certainly not accidental, as it might seem to be from what have been said so far. Positing mathematical objects and being justified in believing them true once that their applicability is discovered, is the right procedure to follow which could turn out not to be successful unless numbers *are* objectively exisitnig objects (as the 18th century mathematicians thought it was the case) or they could be explicated into some objecively existing objects (as we know it is the case with sets).

But then, why should we believe their results, given that they were false? And how come the 18th century mathematics was/is applicable, given that it is false?

Past mathematics being false does not preclude the possibility of its being applicable (and hence useful). It has been false, even though applicable, simply because it does not fulfil the second requirement from the indispensability argument needed for it to justifiably maintain its being true - the indispensability part. Its applicability is not an issue. Numbers are applicable but not indispensable; we can eliminate them in favour of sets. Subsequent to the discovery of the non-existence of numbers, number-theoretic statements are hence to be conceived as general statements about all omega sequences comprising sets.

The semantic functions of numerals have therefore changed through the centuries. As we construe mathematics now is different from how it was construed in the past.

What are numbers then? Numbers are just our posits and talk about numbers is not really about anything determinate. It can be reasonably reconstructed as talk about sets; with many

different reconstructions being possible: two of which the already mentioned Zermelo's and von Neumann's ordinals.

At this point, something bizarre seems to be going on in mathematics though. Viz. in science when our posits turn out not to exist we cease to talk about them. An example might be the case of our positing the existence of the planet "Vulcan" in order to explain the orbit of Mercury - when it turned out that there was not such a planet we stopped speaking of it. Why is it not the case with numbers? Should we not eradicate talk about numbers from the mathematical discourse? Yes, we should. For mathematicians though, discussions about the existence of (mathematical) objects are either marginal or, more often, of no importance. Given that the applicability of numbers is not in jeopardy, they see no good reason for ceasing to speak about numbers. As it stands, it seems that in mathematics we are in fact talking about two things: something that is our posit and something objectively existing but, since it does not demand for any revisionism of the standard mathematical practise, it is of no interest for mathematicians. Numerals therefore partially refer to omega sequences of sets and number theoretical statements are therefore statements about certain sets, where different reductions are possible. What are the rules by which such statements of set theory are mapped into arithmetical statements? What exactly does e.g. "1+2=3" assert? Let us take the von Neumann ordinals and the reduction

$$
\begin{array}{l}
0 \ldots \varnothing \\
1 \ldots \{\varnothing\} \\
2 \ldots \{\varnothing, \{\varnothing\}\} \\
3 \ldots \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
\ldots
\end{array}
$$

Every ordinal is hence the set of all those that precede it. To sum two numerals is, by definition, the set which is an ordinal and which is determined by the "ordered union" of the corresponding sets. Let us λ, μ be two ordinals (in our specific example $\{ \varnothing \}$ and $\{ \varnothing , \{\varnothing\} \}$). We first define the union of disjointed ordered sets (we cannot simply take λ and μ because they are not disjointed) :

$(\lambda \times \{ \varnothing \}) \cup (\mu \times \{\{ \varnothing \}\})$ and then define the sum $\lambda + \mu$ as:

$$(*) \qquad \lambda + \mu = \mathrm{ord}\,((\lambda \times \{ \varnothing \}) \cup (\mu \times \{\{ \varnothing \}\}).$$

The sum is the ordinal number of this union and it comes out to be, in the sequence above, the ordinal $\{ \varnothing , \{\varnothing\}, \{ \varnothing , \{\varnothing\} \} \}$

In this way, the set-theoretic statement (*) get mapped onto "1+2=3".

In summary, given the indeterminacy problem, it becomes clear that numbers do not (objectively) exist and that various proposed reductions to objectively existing sets are equally viable.

Such a solution of the indeterminacy problem might seem to be very similar to Quine's way of dealing with the problem. Quine namely solves the indeterminacy problem by denying the idea that, if numerals refer to objects, then there are particular objects/sets to which they refer. He holds that there are no sets to which numerals absolutely refer and that

> what numbers themselves are is in no evident way different from just dropping numbers and assigning to arithmetic one or another new model, say in set theory.[237]

Quine's view is nevertheless different from the one I suggest on more then one points. Firstly, Quine's underlying theory is the one of the ontological relativity and the idea that:

[237] 'Ontological Relativity' in Quine (1969), p. 44.

> Ontology is indeed doubly relative. Specifying the universe of a theory makes sense only relative to some background theory, and only relative to some choice of a manual of translation of one theory into the other.[238]

His holding that "there is no saying absolutely what the numbers are; there is only arithmetic" is closely related to the basic *ante rem* structuralist's tenet, that is the idea that at least in mathematics one should think of "object" as elliptical for "object of a theory"and that the idea of a single universe, divided into objects a priori, has to be rejected - as it is maintained by e.g. Shapiro.

In his sense, according to Quine, the mathematical case is analogous to the physical one: in the same way in which is impossible to talk about absolute space or velocity, "there is no absolute sense in speaking of the ontology of a theory". This certainly contrasts with platonism (even with the structuralist one concerning structures) and is just remotely related with the idea I was trying to defend. I am holding sets to be objectively, 'absolutely' existing objects, whose existence is independent of the existence of omega sequences.

Besides, Quine ontological relativity precludes the possibility of a correspondence theory of truth and hence of a correspondence relation between words and the extralinguistic world. Quine's view would therefore, if true, mean in a way the platonism's demise.

[238] 'Ontological Relativity' in Quine (1969), pp. 54-55.

11

A Concluding Summary

The goal of this book has been to advocate platonism in the philosophy of mathematics. I have tried to explain in detail what platonism is, what problems it has to deal with, what versions on the contemporary scene I find to be most appealing, which problems these versions still leave unsolved, and how I see some of these problems could be solved.

For the mathematicians who accept standard mathematics, as well as for all those who use it, platonism has to be the underlying philosophical view. There simply is no other choice. Anyone who accepts the axioms and theorems of standard mathematics ought to embrace platonism.

Why? Well, let us take just one example, the axiom of infinity. The number theoretic version of this axiom says that for every natural number, there *exists* a number that is bigger. In set theory, the axiom appears as follows:

$$(\exists M)\,(\varnothing \in M \wedge (\forall A)(A \in M \Rightarrow A \cup \{A\} \in M))$$

That is, there *exists* (at least) one set M with the following property:

1. The empty set is its element, or in symbols: $\varnothing \in M$,
2. If any set A is its element, then the set $A \cup \{A\}$ is its element too.

In set theory, the axiom of infinity says there *exists* an *infinite* set. But an (actually) infinite set certainly cannot be the product of our mind, or a result of our constructions. Therefore, if we want to accept the axiom and take it to be true in some non-trivial sense of the term, there is no other option than to accept platonism. Similar examples are legion.

According to Dummett, platonism could be easily rejected if it did not have followers like Frege or Gödel.[239] But I have argued that this viewpoint is completely misguided. Contrary to what Dummett says, there are many reasons for accepting platonism. Moreover, without good reasons, platonism certainly would not have great mathematicians and logicians among its followers. As Moschovakis says:

> The main point in favour of the realistic approach to mathematics is the instinctive certainty of almost everybody who has ever tried to solve a problem that he is thinking about 'real objects', whether they are sets, numbers, or whatever... [240]

Apart from this, what are these main reasons for endorsing platonism?

Firstly, platonism is in keeping with the fact that the axioms and simple theorems of mathematics are obvious to us. The obviousness, for example, of 2+3=5, is not the result of any laboratory observation, or empiristically tested fact. If in adding two books to three books we obtained six books we would be convinced that we made a mistake while adding them (and of course, we would be right). We certainly would not conclude that 2+3=6. Secondly, platonism is perfectly in keeping with the applicability of mathematics. According to some platonists it is the main *pro* platonism argument, the applicability differs mathematics from any other game. As Frege says in his *Grundgesetze*, vol. II, sec. 89:

> If we stay within [the] boundaries [of formal mathematics], its rules appear as arbitrary as

[239] See his 'Platonism' in Dummett (1978), p. 202.

[240] Moschovakis (1980), p. 605.

> those of chess. [But] applicability cannot be an accident.

Admittedly, platonism is not without problems. Around the turn of the 20th century, the crisis in the foundations of mathematics created by the paradoxes – and by Cantor's set theory too – has eventually spawned several anti-platonist philosophies of mathematics, ranging from "faint of heart" realism to the variety of anti-realisms, such as: intuitionism, according to which a mathematical theorem expresses just a successfully realised mental (Heyting, Brouwer) or logical-linguistic (Dummett) construction; nominalism, which denies the existence of *any* abstract objects, and therefore of mathematical objects as well; and formalism, the goal of which is to solve the problem of paradoxes, and demonstrate the formal consistency of non-finitary mathematics. In this book I have presented their main arguments against platonism, and rebut them.

However, I have tried to do more than advocate platonism. For platonism is not homogeneous. Different versions of it arise depending on how the problem of grasping mathematical objects is viewed, and on which mathematical objects are held to exist. Some platonists endorse the view that there is a platonistic intuition that allows us to grasp the basic mathematical objects and theorems. Hardy[241] thinks we actually "see" certain mathematical results in the same way in which a geologist sees a mountain. Similarly, Gödel thinks that there must be a centre responsible for the perception of sets located near the neuronic speech centre, and that we grasp sets with a perception analogous to the sense perception. As he says in a famous passage from 'What is Cantor's continuum problem':

> . . . the objects of transfinite set theory, ..., clearly do not belong to the physical world and even

[241] Hardy (1948).

> their indirect connection with the physical world is very loose ....
> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretic paradoxes are hardly more troublesome for mathematics than deceptions of the sense are for physics. [242]

However, many platonists disagree with Gödel on this matter. Logicism and neo-logicism, for example, denies the existence of the kind of intuition Gödel had in mind, and claims instead that our mathematical knowledge is based on our capacity to grasp mathematical objects by the specifically *reasoning* faculties of the mind.

In this book, I have confined my attention to those contemporary versions of platonism I find most attractive. These are structuralism, some versions of which are platonistic, and neo-logicism, the doctrine which arises from attempts to preserve the fundamental features of Frege's logicism.

Many problems still remain open and unsettled and I am aware that many of my attempted solutions might seem unsatisfactory and incomplete. The issues I have addressed certainly warrant further study.

[242] In Benacerraf and Putnam (1983), pp. 483-4.

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Platonism in the philosophy of mathematics is the view that (at least some) mathematical objects (such as numbers, sets, functions, etc.) are abstract objects that exist, independently of our constructions and beliefs; and that the mathematical statements we take to be true are, by and large, true.

However, platonism is not homogeneous. Different versions of it arise depending on how the problem of grasping mathematical objects is viewed, and on which mathematical objects are held to exist.

This book tries to explain in detail what mathematical platonism is, what problems it has to deal with, what versions on the contemporary scene are the most appealing ones, which problems these versions still leave unsolved, and how these problems could be solved.

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